1 Introduction

Annex 2

Time: Towards a Consistent Theory, chp. 5b

CHAPTER IV ended with the hope that the introduction of the field may help to resolve the paradoxes of thermodynamics. But in Chapter VA we saw that the introduction of the field seems to create more problems than it solves.

Let us, for the moment, forget about the field, or else let us adopt the point of view that the field is a dispensable intermediary between particles. Let us look, instead, at the nature of the many-body problem (of electrodynamics), and the novel features arising from the *finite speed of interaction*: to measure time, or anything else, one must postulate that the speed of light is constant.

What are the consequences of the finite speed of interaction? We recall Poincaré's remark: '*The state of the world will depend not only on the moment just preceding, but on much older states.*' In both cases we obtain what Poincaré called (p 65) '*equations of finite differences*'. As Poincaré further argued, the substance of physics lies in its mathematical formalism — the 'mechanical explanations' are redundant. So, between fields and particles, it matters little which mental picture we feel comfortable with.

Poincare on FDE

How does the finiteness of the speed of interaction affect the underlying equations? To see this, consider a system of n particles. In the field picture, an accelerated charged particle e_1 gives out retarded radiation which is incident upon other charged particles e_2, e_3, ..., e_n. The outgoing retarded wave accelerates other charged particles at *later* or retarded times. In the particle picture, only these accelerations matter: the acceleration of e_1 at time t depends upon the acceleration of e_2, e_3, ..., e_n, at past or retarded times, say $t-\tau_2$, ..., $t-\tau_n$. Ignoring, for the moment, the question of self-action and radiation damping, the difference between the field-picture and the particle-picture does *not* show up mathematically.

2 The two-body problem of electrodynamics

2. 1 Formulation

For example, the equations of motion of two charged particles i and j, in *one* dimension, interacting solely through retarded radiation (and without radiative damping) take the form:[1]

$$\ddot{z}_i(s_i) = \frac{k}{m}\,\frac{[1+\dot{z}_i^2(s_i)]^{1/2}\ sgn\,[z_i(s_i)-z_j(s_j)]}{\left\{[z_i(s_i)-z_j(s_j)]\dot{z}_j(s_j)-\left|z_i(s_i)-z_j(s_j)\right|c\dot{t}_j(s_j)\right\}^2}\ , \tag{1}$$

where (t, z), with the appropriate subscript, denote the coordinates of the world-lines of the particles, dots denote differentiation with respect to the proper times s_i, s_j of the two particles, and (i,j) = (1,2) or (2,1). Given s_i, the *retarded proper time* s_j in the above equation, corresponds to the point at which the backward null cone from the point $(t_i(s_i), z_i(s_i))$ meets the world line of particle j. This is obtained from (see Fig. 5):

$$c\Big[t_i(s_i)-t_j(s_j)\Big] = \left|z_i(s_i)-z_j(s_j)\right|. \tag{2}$$

The difficulty of having two independent variables can be removed by rewriting (1) as

$$\frac{v'_i(t)}{[\,1-v^2_i(t)/c^2\,]^{3/2}} = \frac{(-1)^i\,k}{m_i\tau^2_{ji}(t)}\cdot\frac{c-(-1)^j v_j(t-\tau_{ji}(t))}{c+(-1)^j v_j(t-\tau_{ji}(t))}, \tag{1′}$$

where $k = -e_1\,e_2/c^2$, the explicit retardations

$$\tau_{ji}(t) = t - t_j(s_j) \tag{2′}$$

are obtained from equation (2), primes denote differentiation with respect to t, and $v_i = z'_i$ is the velocity.

Reforming the notation (to use bars rather than subscripts), and using units with $c = 1$, these equations may be rewritten:

$$\frac{v'(t)}{[\,1-v^2(t)\,]^{3/2}} = \frac{b}{\tau^2}\cdot\frac{1+\bar{v}(t-\tau)}{1-\bar{v}(t-\tau)}\,, \tag{3a}$$

$$\frac{\bar{v}'(t)}{[\,1-\bar{v}^2(t)\,]^{3/2}} = -\frac{\bar{b}}{\bar{\tau}^2}\cdot\frac{1-v(t-\bar{\tau})}{1+v(t-\bar{\tau})}\,, \tag{3b}$$

where $b = -k/m$, $\bar{b} = -k/\bar{m}$, and

$$\tau(t) = |x(t) - \bar{x}(t-\tau)|, \tag{4a}$$

$$\bar{\tau}(t) = |\bar{x}(t) - x(t-\bar{\tau})|. \tag{4b}$$

We can now see the difference quite clearly: *the character of the differential equation has changed.* The system of equations (3) is no longer a system of simple differential equations, but is a system of 'difference-differential equations' or a system of ordinary differential equations (o.d.e.'s) with *retarded deviating arguments.* The 'accelerations' of the particles at time t depend upon their velocities v, $\bar{v}$ at past or retarded times, $t-\tau$, $t-\bar{\tau}$.

What is the significance of this change? To study the two-body problem of electrodynamics, one must study o.d.e.'s with deviating arguments: even the simplest qualitative features of such o.d.e.'s completely destroy the Newtonian paradigm, and suggest a resolution of the paradoxes of thermodynamics, and the paradoxes of advanced action.

Effect on Newtonian paradigm

2. 2 Some definitions

A first-order o.d.e. has the form

$$x'(t) = f(t, x(t)), \tag{5}$$

where f is some function (generally non-linear). It is well known that the most general system of o.d.e.'s can be reduced to a system of such equations, i.e., to the form (5), regarding x as a vector, if necessary. It is also well known that (under some mild requirement of continuity on f) prescription of the 'initial value' $x(0)$, at an instant $t = 0$, determines a unique solution $x(t)$ of (5) in a neighbourhood $[-\delta, \delta]$ of $t = 0$. This is the Newtonian paradigm.

Newtonian paradigm defined.

In contrast, a differential equation with deviating arguments has the form

$$x' = f(t, x(t), x(t-\tau)). \tag{7}$$

The 'dependent variable', the function x, appears for more than one value of its argument, the 'independent variable', t.

An equation with deviating arguments is classified as *retarded*, or *history dependent*, if the highest order derivative of the unknown function appears for exactly one value of the argument, and this argument is not less than all the arguments of the unknown function and its derivatives appearing in the equation. For example,

$$x'(t) = f(t, x(t), x(t-\tau(t))) \tag{8}$$

is called retarded if $\tau(t)>0$.

Similarly, the equations of motion of charged particles interacting solely through advanced radiation correspond to *anticipatory* behaviour, or an o.d.e. with *advanced* deviating arguments,

$$x'(t) = f(t, x(t), x(t+\tau(t))). \tag{8}$$

The definition requires the same proviso as above, except that now the argument of the highest order derivative must be less than or equal to all the other arguments of the unknown function, i.e., $\tau(t)>0$ in (8), or $\tau(t)<0$ in (7).

More generally, a system of charged particles interacting through both advanced and retarded radiation displays *partly anticipatory* behavior, corresponding to an o.d.e. with *mixed-type* deviating arguments:

$$x'(t) = f\,(t, x(t), x(t-\tau_1(t)), x(t+\tau_2(t)))\,. \tag{9}$$

From now on, equations of the type (7), (8), and (9) will be referred to as retarded, advanced, and mixed-type o.d.e.'s. The mathematical theory of such differential equations with deviating arguments, also known as functional differential equations, differs from the mathematical theory of the usual differential equations.

2. 3 The recurrence paradox and the past-value problem

Van Dam and Wigner[2] considered equations involving both retarded and advanced fields. They asserted (without proof) that instantaneous positions and velocities were sufficient to determine unique trajectories.

Now, with electromagnetic interactions taken into account, the many-body equations of motion (3) are retarded o.d.e.'s. However, for the simplest model of even a retarded differential equation, modeling a history-dependent situation, it is inadequate, in general, to provide initial data at a point. Consider, for instance, the o.d.e with constant retardation $\pi/2$,

$$x'(t) = x(t-\frac{\pi}{2}), \quad t \geq 0\,. \tag{10}$$

To obtain a unique solution, it is insufficient to specify only the state at one point of time, say $x(0)$. Thus, $x = \cos t$ and $x = \sin t$ are obvious solutions, and since the equation is linear $x = a\cdot\cos t + b\cdot\sin t$ is a solution for arbitrary constants a and b, and both a and b cannot be determined from a knowledge of $x(0)$.

Past history is required

Since the behavior is history-dependent, it is more reasonable to ask for a unique solution after prescribing the *past history*, i.e., an *initial function* $x = \varphi$, over the relevant part of the past: the interval of retardation, $[-\pi/2, 0]$.

In general, a unique solution of the past-value problem for the retarded system,

$$x'(t) = f\,(t, x\,(\,t-\tau_1(t), t-\tau_2(t), ..., t-\tau_n(t))\,)\,, \tag{11}$$

may be obtained under the following sufficient conditions.[3] (i) All delays, τ_i, are bounded, and (ii) some technical conditions such as a local Lipshitz condition and a continuity condition are satisfied.

From the point of view of thermodynamics, the interesting conclusion is the following. The hypotheses underlying the recurrence paradox have been destroyed: *there is no longer a unique trajectory through each point of phase space*. More than one trajectory may pass through each point of phase space; trajectories may intersect (Fig. 1).

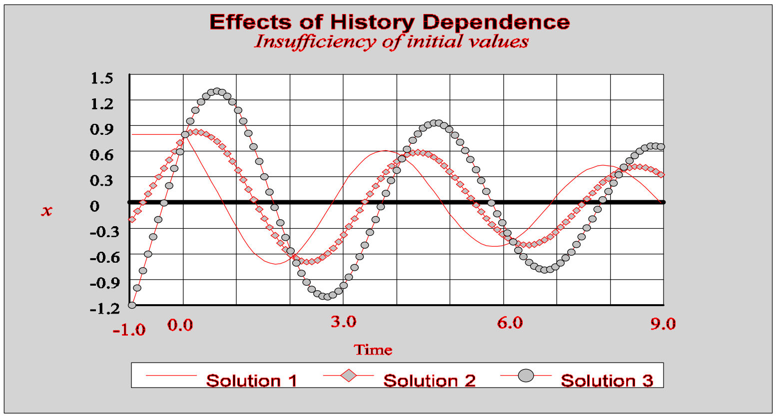

Fig. 1: Effects of history-dependence

Three solutions of a model first-order delay equation, showing the effects of history-dependence. All three solutions have the same initial value x(0), though the past histories prescribed over the time-interval [-1, 0] are different. Note also the discontinuity in the derivative of solution 1, visible at t=0.

2. 4 The reversibility paradox: time asymmetry of delay

For retarded o.d.e.'s the intersection of trajectories takes place preferentially towards the future, in a way that destroys the hypothesis underlying the reversibility paradox. An ordinary differential equation is time symmetric: it may be solved either forward or backward in time. From a knowledge of the current values, Newton's laws may be used to predict the future or retrodict the past. However, a retarded o.d.e. relates past causes to current effects. Such an equation may be solved forward in time, but not, in general, backwards in time.

Consider the following ordinary, linear, retarded differential equation with constant coefficients, and constant retardation r:

$$x'(t) \;=\; a\,x(t) + b\,x(t-r), \tag{12}$$

with b different from zero and $r>0$. To solve the equation backwards, it is only necessary to solve an algebraic equation,

$$x(t-r) \;=\; \frac{x'(t)-a\,x(t)}{b}, \tag{13}$$

to obtain the solution on $[t-2r, t-r]$, given $x = \varphi$ on $[-r, 0]$. For nonlinear equations, this already means that backwards solutions will not be unique. For the case under consideration, suppose $a \neq -b$ and we prescribe $\varphi(t) \equiv k$, a constant, on $[-r, 0]$, and ask for a backwards solution for $t \leq 0$. Then (13) implies that $x(t) = -ak/b$ so that the unique solution of the algebraic equation (13) fails to be continuous, and hence differentiable. Therefore, a (continuous) backwards solution of (12) does not exist in general.

Of course, one could think of choosing a final function in such a way that the solution exists. But then the solution would, in general, fail to be unique. Consider

$$x'(t) \;=\; b(t)\,x(t-1), \tag{14}$$

where b is any sufficiently smooth (e.g. continuous) function which vanishes outside [0, 1], and with

$$\int b(t)\,dt = -1. \tag{15}$$

For example,

$$b(t) = \begin{cases} 0 & t \le 0, \\ -1 + \cos 2\pi t & 0 \le t \le 1, \\ 0 & t \ge 1. \end{cases} \tag{16}$$

For $t \le 0$, (14) reduces to $x'(t) = 0$ so that, for $t \le 0$, $x(t) = k$ for some constant k. Now if k is *any* constant then, for $t \in [0,1]$,

$$x(t) = x(0) + \int_0^t x'(s)\,ds$$

$$= x(0) + \int_0^t b(s)\,x(s-1)\,ds$$

$$= x(0) + x(0)\int_0^t b(s)\,ds, \tag{17}$$

since $x(s-1) \equiv k = x(0)$ on [-1, 0]. Hence, using (15), $x(1) = 0$ no matter what k was. But $x(1) = 0$, and $b(t) = 0$ for $t \ge 1$, implies, by (14), that $x(t) = 0$ for *all* $t \ge 1$. Consequently, (14) does not admit a unique backwards solution even if we prescribe future data for all future times $t \ge 1$. Thus, if φ differs from 0 on $[1,\infty)$ there are no backward solutions. But, if $\varphi \equiv 0$ on $[1, \infty)$, the solutions branch into the past (Fig. 2), and there is no way to pick a unique solution from the infinity of continuous solutions that are available.

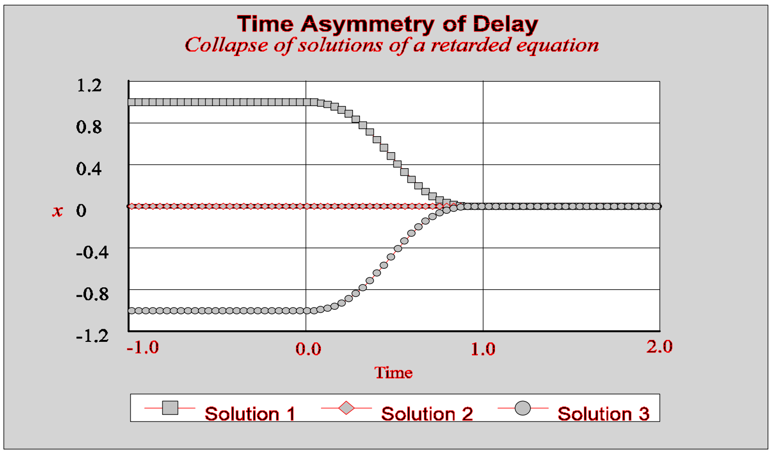

Fig 2: Time asymmetry of delay

Three solutions of the retarded equation (14) which collapse towards the future. The different past histories, prescribed over the time-interval [1,0], all result in the same future solution for t≥1. Retrodiction is hence impossible from future data prescribed over t ≥ 1. Teleological 'explanations' are impossible with history-dependent evolution.

2. 5 Preacceleration: the Taylor-series approximation

We saw in Chapter VA that the study of radiative damping and, in particular, the Schott term, leads to equations that are of the third order in time, resulting in the preacceleration of the electron. Dirac, in 1938, obtained these equations by means of a Taylor expansion which seems unavoidable.[4] Many other authors[5] have attempted similar approximation procedures, using a Taylor series expansion to get rid of retarded/advanced expressions, in dealing with the two-body problem in electrodynamics and gravitation. Physically, this procedure means that we model a history-dependent system by an instantaneous system with additional degrees of freedom.

Einstein's mistake

This procedure is known to be, in general, invalid. This may be seen from the following counter-example,

$$x'(t) = -2\, x(t) + x(t-r), \tag{18}$$

where $r>0$ is a small constant. Every solution of this equation is bounded[6] and tends to zero as $t\to\infty$. But if we choose the Taylor series approximation to the right hand side and truncate after two terms, we obtain

$$x'(t) = -2\, x(t) + [x(t)-rx'(t) + \frac{1}{2} r^2 x''(t)], \tag{19}$$

which admits exponentially increasing solutions $x(t) = c \exp(\alpha t)$, with $\alpha > 0$. Thus, the Taylor approximation of (18) by (19) leads to qualitatively incorrect behaviour, no matter how small r is, so long as $r>0$.

It may be shown that it is not the order of the approximation which is at fault: with instantaneous data, even an infinite number of degrees of freedom is inadequate. The order of the approximation does, however, make a difference from the numerical point of view, as pointed out by El'sgol'ts,[7] 'since the transition is equivalent to the rejection of the term with the highest order derivative in an *unstable-type differential equation* with a small coefficient before the highest derivative.'[Emphasis mine]

In the usual treatment of the numerical solution of retarded o.d.e.'s, attention is focused upon the discontinuities that might arise at the ends of delay intervals (e.g. Solution 1 of Fig. 1). However, one would expect the general electrodynamic many-body problem to be 'stiff': there could be oscillations at widely varying frequencies. In view of the Dahlquist barrier,[8] A-stability fails for any rule higher than the trapezoidal rule, so that the Taylor approximation could be numerically misleading for derivatives of order greater than two.[9] Thus, Dirac was perhaps right in a way when he rejected the higher order terms as too complex to apply to 'a simple thing like the electron'.

To summarize, the origin of the Schott term in the Lorentz-Dirac equation of motion is mathematically dubious, and can result in qualitatively incorrect behaviour, though it may yet provide a more robust numerical approximation than would be obtained by the inclusion of higher-order relativistically covariant terms. The alternatives that have been proposed,[10] to the Lorentz-Dirac equation, have not proved satisfactory.[11]

VIB QUANTUM-MECHANICAL TIME

ABSTRACT. We present a brief exposition of the orthodox axiomatic approach to q.m., indicating the relation to the text-book approach. We explain why the usual axioms force a change of logic. We then explain the attempts to *derive* the Hilbert space and the probability interpretation from a new type of 'and' and 'or' or a new type of 'if' and 'not'. Included are the Birkhoff-von Neumann, Jauch-Piron, and quantum logic approaches, together with an account of their physical and mathematical obscurities.

Instead of entering the labyrinth of subsequent developments, which seek new algebraic structures while accepting the old physical motivation, we present an exposition of the structured-time interpretation of q.m., which seeks a new physical motivation.

We saw in Chapter VB that, with a tilt in the arrow of time, the solutions of the many-body equations of motion are intrinsically non-unique. In Chapter VIA we had indicated how this non-uniqueness relates to a change in the logic of time. We now explain how the resulting changes in the logic and structure of time lead to a new type of 'if' and 'not', of the kind required by q.m., while escaping from the criticism which applies to the earlier 'quantum logic' approaches.

We briefly indicate the analogy between this logic and the temporal logic required for the formal semantics of parallel-processing languages like OCCAM, and distinguish the structured-time interpretation from the superficially similar many-worlds interpretation and the transactional interpretation of q.m.

1 Introduction

THE preceding chapter introduced the problem of a non-trivial structure of time: the (local) topology of time, in the real world, might be different from that of the real line. The real-line topology differs from the mundane view of a past-linear future-branching time, used to demarcate and validate physics. Moreover, there is possible incoherence about the structure of time, even within physics, as different structures may be simultaneously implicit.

We explained how the notion of a structure of time could be formalized in terms of properties of the earlier-later relation (U-calculus) or, more generally, using an appropriate (temporal) logic.

This chapter deals with two earlier claims (a) that an appropriately structured time could be related to the change of logic required by the axiomatic formulation of q.m., and (b) that the hypothesis of a tilt in the arrow of time implies such an appropriate structure. The other consequence of the basic hypothesis, viz. non-locality, is hardly a serious drawback since we saw in Chapter VIA that locality is a fuzzy and metaphysical requirement which lacks a basis even in classical physics.

To reiterate, the aim of this chapter is to present an exposition of the structured-time interpretation of q.m. which relates the many-body equations of motion of non-local classical (relativistic) mechanics, the emergence of a logical structure, or a non-trivial topology of time,[1] and the mathematical formalism of q.m.

§ 2 presents an exposition of the orthodox Hilbert-space axiomatics of q.m. and relates it to the usual textbook approach. § 3 explains why the orthodox axiomatic approach forces a change of logic and goes on to present an exposition of the 'quantum logic' approach, its relation to the Hilbert-space axiomatics, and its obscurities. The idea is to distill the body of q.m. to its algebraic and logical 'skeleton'. Finally, § 4 presents the structured-time approach,