(ii) Title of the project.

A New Method of Teaching the Calculus

(iii) Specific objectives.

To test the effectiveness of a proposed new method of teaching the calculus.

(iv) Methodology adopted.

Phase 1: Development of the detailed teaching modules.

The first step would be to train the teacher trainers with this approach. The feedback will be used to develop the teaching modules. This phase would last approximately one month.

Phase 2. Preliminary survey.

(a) A convenient sample of four nearby schools would be selected.

(b) The students of class XI would be administered a pre-test in two parts. The first part would consist of problems drawn at random from specified calculus texts, and would test the student's conventional skills at calculus, specifically the ability to integrate and differentiate various functions and their combinations. The second part would involve more innovative questions which would test the student's understanding of calculus concepts, and the ability to apply the calculus knowledge to new situations 'beyond the book'

(c) The students' performance in the pre-test would be used to stratify the population, and a stratified random sample would be drawn, representing students at all levels of performance.

Only the first part of the pre-test would be used as a stratification variable, since our expectation is that students would perform poorly in the second part, where scores would be clumped close to zero.

(d) The selected sample of students would be taught for 10 hours spread over two weeks. This teaching would prominently involve (1) basics of a new approach to mathematics as calculation. Then (2) students would be taught calculus as related to the solution of ordinary differential equations. To facilitate this the students would also be taught (3) to use the software CALCODE. To ensure that the students retain the currently taught ability to carry out symbolic manipulation (4) the students would also be taught to use the open source software MAXIMA (earlier MACSYMA).

(e) At the end of this period, the student would be administered another test, similar to the first test. However, they would be allowed to use a computer.

Depending upon the availability of the students, this phase could last approximately 3 months.

Phase 3. Final survey.

Based on our experiences with these four schools, the teaching modules would be appropriately revised, and the process would be repeated using an appropriate random sample of schools.

This phase would be carried out only if sufficient funding is available, or enough schools show an interest.

(v) Number and names of personnel (Teachers/Principal/ Headmaster member of the M.C, parents etc.) participating in the execution of the project.

C. K. Raju, Nidhi Agarwal, Pallavi Mishra, Monika Verma, Chandramani Naik, Ram Naresh, Ahinsa, Gaurav Parashar

(vi) Name of the Project Coordinator.

C. K. Raju

(vii) Likely date of commencement of the project.

1 September 2008

(viii) Likely date of completion of the Project.

1 months from start date for phase 1.

3 months from start date for phase 2.

Depending upon funding and interest for phase 3.

(ix) Background work carried out by the Project Coordinator.

(a) References

Books by C. K. Raju

0. *Cultural Foundations of Mathematics: The Nature of Mathematical Proof and the Transmission of the Calculus from India to Europe in the 16th c. CE.* Pearson Longman, New Delhi 2007. PHISPC, vol X.4.

- *Calculus Without Limits. A New Method of Teaching the Calculus with Rigour, Understanding and Advanced Applications* (in preparation).

Articles and talks on calculus and math education by C. K. Raju

1. "Computers, Mathematics Education, and the Alternative Epistemology of the Calculus in the Yuktibhasa". Invited plenary talk at the 8th East West Conference, Hawai'i. In: *Philosophy East and West* **51**(3) (2001) pp. 325-62. Draft at http://IndianCalculus.info/Hawaii.pdf.

2. "How and Why the Calculus was Imported into Europe". Talk delivered at the International Conference on *Knowledge and East-West Transitions*, National Institute of Advanced Studies, Bangalore, Dec 2000. Abstract at http://IndianCalculus.info/Bangalore.pdf.

3. "The Calculus: its Indian Origins and Transmission to Europe prior to Newton and Leibniz", invited talk, conference, Univ. of Louisiana, Lafayette. Also, Dept of Maths, Univ. of Iowa at Ames, and public lecture with the same title, Oct 2004.

4. "Math Wars and the Epistemic Divide in Mathematics", invited talk at Centre for Research in Mathematics and Science Education, Univ. of San Diego, Oct. 2004. Also, paper presented at International Conference, Episteme-1, Goa, Dec 2004. At http://www.hbcse.tifr.res.in/episteme1/themes/ckraju_finalpaper.

5. "Time: What is it That it can be Measured?" In: *Science&Education*, (Kluwer/Springer BV) **15**(6) (2006) pp. 537–551, DOI 10.1007/s11191-005-5287-z.

6. "Time Measurement in Classical Indian Tradition and the Present-Day Representation of Time as a Continuum", invited plenary talk at the 2nd International Pendulum Program Conference, Univ. of New South Wales, Sydney, 13 Oct 2005. In *Proc. 2nd International Pendulum Conference*, ed. M. R. Matthews, UNSW, Sydney, 2005, pp. 225-248.

7. "The Indian Origins of the Calculus and its Transmission to Europe. Part I: Series Expansions in India from Aryabhata to Yuktidipika." Talk at Dept of Math, University of Auckland, New Zealand, 17 Oct 2005.

8. "The Indian Origins of the Calculus and its Transmission to Europe. Part II. Lessons for Mathematics Education." Talk at University of Auckland, New Zealand, Dept. of Math, 18 Oct 2005.

9. Introductory talk, symposium on "Mathematics in Relation to History, Culture, and Technology", India International Centre, New Delhi, Nov 2007.

10. "Transmission of the Calculus from India to Europe before Newton and Leibniz", invited talk at Indian Institute of Advanced Study, Shimla, Nov 2007.

11. “Good-Bye Euclid!”, paper presented at Indian Social Science Congress, Mumbai, Dec 2007, session on mathematics education.

12. “The Indian Rope Trick”, ISSA Congress, Mumbai, Dec. 2007, session on mathematics education.

13. “Two plus two is not always four: Why zero so confused Europeans”, invited talk at International Seminar on Philosophy of Science, Vishwabharati, Shantiniketan, Jan 2008.

Other closely related publications by C. K. Raju:

14. “Mathematics and Culture”, in *History, Culture and Truth: Essays Presented to D. P. Chattopadhyaya*, Daya Krishna and K. Satchidananda Murthy (eds), Kalki Prakash, New Delhi, 1999, pp. 179–193. Reprinted in *Philosophy of Mathematics Education* **11**. Available at www.ex.ac.uk/~PErnest/pome11/art18.htm.
15. “Some Remarks on the Indian Epistemology of Mathematics.” Talk delivered at the International Laboratory for the History of Science, *The Material Culture of Calculation*, Max Planck Institute, Berlin, June, 1999.
16. “How Should ‘Euclidean’ Geometry be Taught?” in Nagarjuna G. (ed) *History and Philosophy of Science: Implications for Science Education*, Homi Bhabha Centre, Bombay, 2001, pp. 241–260.
17. “The Religious Roots of Mathematics”, *Theory, Culture & Society* **23**(1–2) Jan-March 2006, Spl. Issue ed. Mike Featherstone, Couze Venn, Ryan Bishop, and John Phillips, pp. 95–97. (Based on a talk delivered at the National University of Singapore, Oct 2005.)
18. “The Mathematical Epistemology of Sunya,” in: *The Concept of Sunya*, ed. A. K. Bag and S. R. Sarma, Indian National Science Academy, Indira Gandhi National Centre for the Arts, and Aryan Books International, New Delhi, 2002, pp. 168–181.
19. “Logic”. Article for *Encyclopaedia of Non-Western Science, Technology and Medicine*, Springer, 2008.

Related Reports in Newspaper and Magazines

- “Mathematics and Culture: Implications of philosophy and culture for contemporary mathematics.” Review of *Cultural Foundations of Mathematics* in *The Hindu*, by K. Srinivasa Rao, 12 Feb 2008, p. 17.
- “Cultural Foundations of Mathematics.” In Book Review Section, *Ghadar Jari Hai* **2**(1) (2007) pp. 26-29.
- “HT correction” *Hindustan Times*, New Delhi, 25 Aug 2007, p. 2. (“The claim made by two British researchers that they were the ones who unearthed the fact the Kerala mathematicians invented the calculus long before Isaac Newton (HT, August 14, 2007, p. 1) was incorrect. ...Recent work on

transmission of the calculus was first done by C. K. Raju...") Corrections also published by some other newspapers. News article by Roger Highfield (Science Editor) withdrawn from website of *Telegraph* (London).

- "Prof Raju's charge of plagiarism found correct: UK varsity warns lecturer". *Hindustan Times*, Bhopal Live, 8 Nov 2004, front page headline.
- "Raju plans to demystify 'sudden spate of European inventions' ", *The Hindustan Times*, Bhopal, page 2, headline, 15 Sep 2004. Also, first part of same interview, 14 Sep 2004, page 3 headline.
- Etc.

(b) Summary

Clearly it is difficult to summarise all the above work. But some brief points are as follows. (Numbers in brackets are references to the books and papers listed above.)

1. Mathematics is *not* universal, but has been understood differently in different cultures.

   For example, the very word "mathematics" derives from the Greek word "mathesiz" which means "learning". Socrates explains (Plato, *Meno*) that learning is the process of the soul recollecting its eternal ideas, and mathematics is specially useful for this, since the soul is stirred by the eternal truths of mathematics. Plato, in his *Republic*, recommended the teaching of mathematics and music for the good of the soul, and not for its practical applications.[0, 1] These religious considerations are no longer part of current mathematics, showing that the understanding of mathematics has changed with time.

2. The present-day idea that mathematics means theorem-proving involves religious beliefs.

   While the concerns of Neoplatonists regarding mathematics have been rejected, these have been replaced by another set of religious considerations. Western mathematics still suffers from a religious hangover in thinking that mathematics involves necessary truths (propositions true for all time or true in all possible worlds). It is this religious hangover (that logic binds God who created the universe) which has led to the current belief in the universality of mathematics.[16] Logic is not universal; for example, Buddhist and Jaina logic are different from the logic currently used in mathematical proof. [0, 1, 14, 17, 18, 19] Using a different logic would change the theorems of mathematics. The decision about which logic to use is a purely cultural decision. If a decision regarding logic were to be taken empirically, then mathematical proof too could involve the empirical. (This is not permitted today, since empirical or inductive proofs are believed to be contingent truth, and treated as weaker than deductive proofs.) In any case, empirically, quantum logic is not 2-valued.[1] Thus, theorem-proving can lead only to cultural truth, and is a completely useless activity for people from other non-Western cultures who do not share those cultural beliefs.

3. Traditional Indian mathematics was oriented towards calculation: it was secular and had practical aims.

   In India, arithmetic was practically oriented, towards commercial tasks, while the calculus related to the building of planetary models (which were used for the practical purpose of making

1 In this connection, see also C. K. Raju, "Quantum-Mechanical Time", Physics Education 10(2) (1993) 143-161, and chapter 6b of C. K. Raju, *Time: Towards a Consistent Theory* (Kluwer Academic, Dordrecht, 1994; Fundamental Theories of Physics, vol. 65). Also, "Some Remarks on Ontology and Logic in Buddhism, Jainism and Quantum Mechanics." Invited talk at the conference on *Science et engagement ontologique*, Barbizon, October, 1999.

a calendar, which is a key technology for monsoon-driven agriculture). Mathematical proof was like any other kind of proof, and it used the empirical, which was not regarded as inferior. Consequently, there was no conceptual difficulty in measuring the length of a curved line by using a rope, as in the *sulba sutra*-s.

4. Indian arithmetic and calculus travelled to Europe (for its practical value) but Europeans tried to assimilate it within their religious understanding of mathematics.

This traditional Indian mathematics (arithmetic, calculus) travelled to Europe [0, 1, 2, 4] because it was valued for its applications (arithmetic for commerce, and calculus for applications to the European navigation problem, and planetary models). However, the imported mathematics was sought to be reinterpreted in terms of the semi-religious understanding of mathematics prevailing in Europe, and inherited from Egypt and Greece. Apart from believing it to be universal, Europeans saw mathematics as perfect, infallible and metaphysical. In the case of arithmetic this led to the problems in Europe, between the algorismus and abacus techniques, regarding the notion of number, particularly zero.[0, 13] In the case of the calculus, which was picked up by Fermat and Pascal, Descartes declared it beyond the capacity of the human mind to determine the length of a curved line. Galileo concurred, and left this 'disreputable' subject to his student Cavalieri. [0, 12]

5. The present-day difficulties in learning mathematics arise from these European difficulties with mathematics. On the principle that phylogeny is ontogeny, the historical difficulties are played out in fast-forward mode in the classroom.

The present-day teaching of mathematics follows the historical sequence in Europe. For example, the calculus is taught in class 11 without defining the notion of derivative, and without defining the notion of limit.[2] (This was the situation prevailing in Europe in the 17th and 18th c.) To define these notions requires formal real numbers, which are the subject of courses on "advanced calculus" or "real analysis".[3] (Formal numbers were introduced in Europe by Dedekind in the 19th c.) Defining formal real numbers involves supertasks[0, 1, 6] which are pushed into the domain of set theory. However, the student is taught only naïve set theory, with all its confusions,[4] but not axiomatic set theory.[5] which is left for more advanced courses. (Axiomatic set theory evolved in Europe only in the 20th c.). Most students never take that advanced course, and thus remain confused about mathematics all their lives. The whole process has no practical value whatsoever, and leaves the students befuddled because of the demand for epistemological satisfaction according to Western European religious beliefs.

6. The solution, therefore, is to de-theologise mathematics, and to go back to the context in which the mathematics originally developed.

According to my analysis [0, 1], the calculus developed with Arybhata's numerical solution of a difference equation for the sine function. This technique is similar to what is today called the Euler method of solving ordinary differential equations (ODEs). Thus, the idea is to teach calculus as the numerical solution of ordinary differential (or difference) equations.

7. This solution suits the idea of mathematics as a secular subject taught for its practical applications using computers.

This idea of teaching the calculus as the numerical solution of ODEs suits the aims of teaching mathematics for its practical applications. By way of preparation this requires a new approach to

2 For example, the current NCERT text states: "First, we give an intuitive idea of derivative (without actually defining it). Then we give a naive definition of limit and study some algebra of limits". J. V. Narlikar et al. *Mathematics: Textbook for Class XI*, NCERT, New Delhi, 2006, chp. 13 "Limits and Derivatives"', p. 281.

3 e.g. D. V. Widder, *Advanced Calculus,* 2nd ed., Prentice Hall, New Delhi, 1999. W. Rudin, *Principles of Mathematical Analysis*, McGraw Hill, New York, 1964.

4 e.g. P. R. Halmos, *Naive Set Theory*, East-West Press, New Delhi, 1972.

5 e.g. L. Mendelson, *Introduction to Mathematical Logic,* van Nostrand Reinhold, New York, 1964.

mathematics as calculation, and a new understanding of number. (E.g. Floating point numbers on a computer do not obey the same "laws" as formal real numbers, since the associative law, for example, fails.) While hand calculation would be taught, naturally computers would be used to facilitate the process. Computer-literate students can easily write their own simple code. Further, this author has developed a software CALCODE for this purpose of solving and visualising the solution of ODEs [5]. The other software that would be used is the open source software MAXIMA (earlier called MACSYMA) for symbolic manipulation. With the aid of this the students will still be able to carry out all symbolic manipulation tasks that they are taught to perform (without understanding) under the present method of teaching calculus. However, they will now have an improved understanding of what they are doing.

8. This approach involves no supertasks, is easy to understand, and allows treatment of advanced topics normally omitted in a calculus course.

In this approach one works with a finite but flexible number system (similar to floating point numbers on a computer), so the student is not forced to learn off promises of future clarification. Instead of derivatives, one has difference quotients which arise naturally (by the rule of 3). Instead of limits one has the better approach via zeroism [0, 18] that was used in traditional Indian mathematics, and Buddhist philosophy. Instead of set-theoretic formal functions one has tables with concrete interpolation procedures. Students can easily understand advanced concepts. For example, the exponential function is not defined in usual calculus courses (since its definition requires advanced concepts like uniform convergence). However, it is easy to define as the solution of the ODE $y' = y$, with $y(0) = 1$. This approach allows advanced applications to physics, such as the motion of the pendulum via Jacobian elliptic functions, or ballistics with air resistance etc.[5]

(x) Level of School Education/Teacher Education-Pre primary/ Primary/Middle/Secondary/Higher Secondary.

Higher Secondary.

(xi) Likely outcome and benefits.

The students will understand and enjoy calculus instead of dreading it, as has been happening for centuries in Europe. Because of the ease of learning, they will be able to learn much more than is taught in a normal calculus course. They will be able to apply this knowledge to more advanced problems of physics. Side by side they will also learn to use symbolic manipulation software to do in a jiffy the toughest exercise in a current calculus texts (which exercise they are currently expected to carry out without understanding the basic notion of derivative and integral). Since the program is open source, it can be modified to print out the rules used in arriving at the answer. Thus, students do not have to learn a skill which is largely useless today, and can instead focus on more

(xii) Time schedule.

Given above. Phase 1: 1 month. Phase 2: Approximately 3 months. Phase 3: Depending upon response and availability of resources for research.