Step 1. Construct a square with sides equal to the diameter of the required circle.

Step 2. Draw the north–south and east–west lines to form four small squares. The required circle meets the square at the four cardinal points. Draw a line from the centre to the south-east corner.

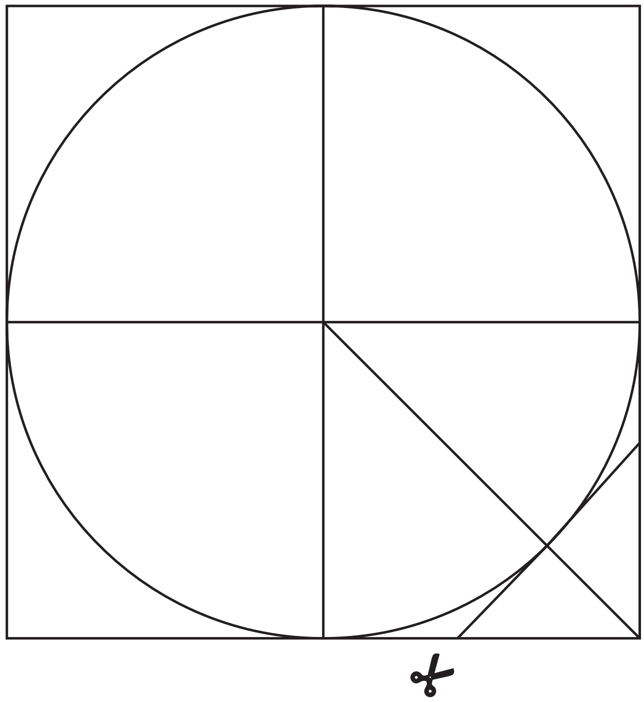

Figure 3.1: **Cutting corners.** The desired circle is the one inscribed in the polygon. At each stage one cuts off from the corner of the polygon an isosceles triangle by measuring out the sides, from the corner. The base of the triangle is tangential to the desired circle.

Step 3. The idea is to cut the south-east corner C along the line AB, and to repeat this process at the remaining 3 corners of the square. The requirement is that the resulting octagon (Fig. 3.2) should be equilateral. Alternatively, the requirement is that the line AB should be tangential to the required circle at the point where the circle intersects the line OC from the centre to the south-east corner.
Step 4. Let x be the side of the required octagon, and r be the radius of the required circle. Applying the sine rule to the right-angled isosceles triangle ABC with hypotenuse AB, we obtain the quadratic equation $x^2 = 2\left(r - \frac{x}{2}\right)^2$, with positive root $x = 2r(\sqrt{2} - 1) = 2(h - r)$, where $h = \sqrt{2}r$ is the diagonal of the smaller square.
Step 5. Since the triangle ESC is similar to triangle ABC, $\frac{h}{r} = \frac{x}{r - x/2}$, so by the rule of three $r - \frac{x}{2} = \frac{rx}{h}$. Measure out this last quantity ($= CA,\ CB$, Fig. 3.2) and cut the corner. (Observe that this quantity corresponds to an irrational number, that is being calculated and measured out, a process inconceivable in the synthetic reinterpretation of "Euclidean" geometry.)
Step 6. The first approximation to the circumference ($= 2\pi r$) is $8x$, and this gives $\pi \approx 3.313708$.
Step 7. (Fig. 3.3) The idea is to cut the corner B of the octagon, along the line B_1B_2, and to repeat this at the other seven corners, to get a 16-sided figure. Observe that the required circle meets each polygon tangentially at the mid-point of its sides. Thus, the line joining

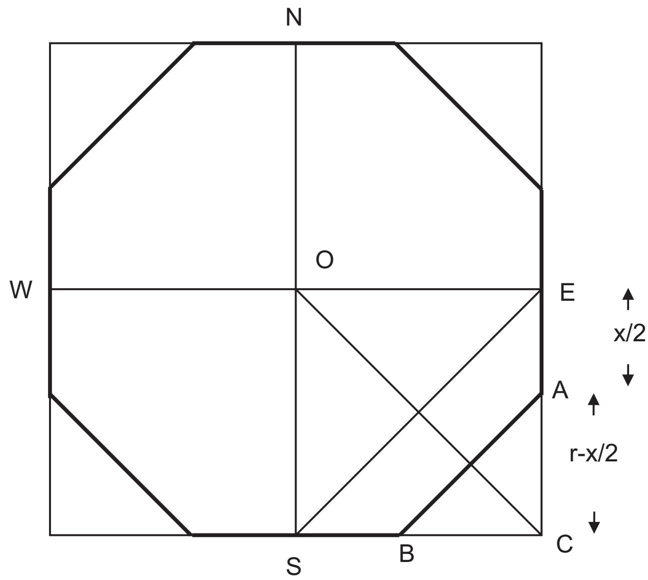

Figure 3.2: **The octagon method.** This method of calculating circumference or π starts with a square of side equal to the diameter of the desired circle, and proceeds by cutting off the corner of the square and of the successive polygons so obtained at each stage, to obtain the next equilateral polygon. This differs from the hexagon-doubling method attributed to Archimedes and Liu Hui.

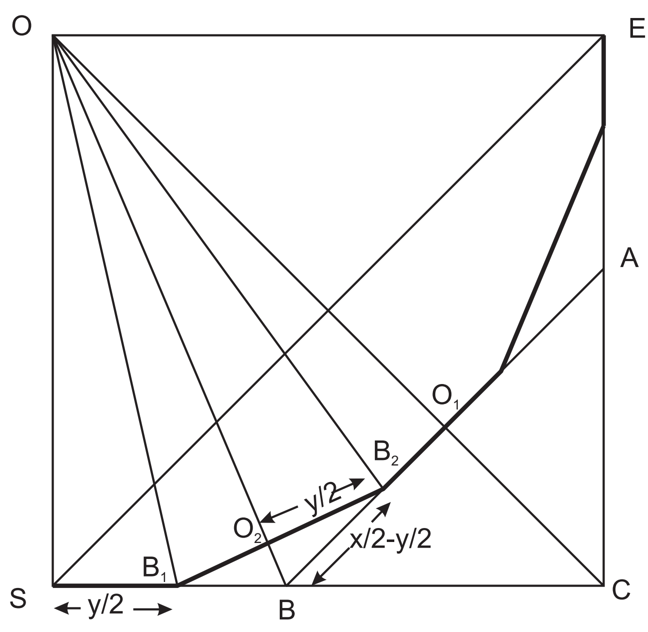

Figure 3.3: **Detail of the octagon-doubling method.** The figure shows the situation in the south-east square where the two corners of the octagon at B and A are cut by calculating and measuring out the sides of an isosceles triangle. The key to the recursion formula is that the required circle meets each such polygon tangentially at the mid-point of its sides.

the centre to the mid-point of the side of the octagon has length r. Solving the right-angled triangle OBO_1, gives $(OB)^2 = r^2 + \frac{x^2}{4}$, hence $BO_2 = OB - r$. But $BB_2 = \frac{x}{2} - \frac{y}{2}$, and $O_2B_2 = \frac{y}{2}$, so we can calculate y by applying the sine rule to the triangle BO_2B_2. In fact, this gives the formula $y = r\frac{a^2-k^2}{a}$, where $a = \sqrt{2} - 1$, $k = \sqrt{(a + a^2)} - 1$, and $\pi = 16\frac{k}{a}$.

Step 8. The method and calculations in the above step can be repeated indefinitely. Hence, we are led to the following numerical algorithm. Let

$$g(x) = \sqrt{(1 + x^2)} - 1,$$

$$f(x) = \frac{g(x)}{x}.$$

The algorithm computes, to level n,

$$z_0 = a = (\sqrt{2} - 1),$$

$$z_i = f(z_{i-1}),$$

$$\pi \approx 2^{2+i+1} z_i.$$

It is clear that the algorithm involves computation of only squares and square roots, and Āryabhaṭa had already stated efficient algorithms for these, which use the decimal place value notation. We took a short cut, and wrote a computer program, using the intrinsic `sqrt` function in Turbo C. The results show that Āryabhaṭa used either the value $n = 5$, or the value $n = 6$, corresponding to a polygon with 512 sides or 1024 sides. In particular, Āryabhaṭa's octagon method could *not* have been the method used by Liu Hui, who clearly used a technique similar to that of "Archimedes", since $3072 = 3 \times 1024 = 3 \times 2^{10}$ is not a power of 2 but is a number that would be obtained on the hexagon-doubling method. The same method of hexagon-doubling must have been used by al-Kashi, since he used a polygon with 3×2^{28} sides.

V
THE DERIVATION OF THE SERIES EXPANSION

Computation of the Circumference

Having outlined the above procedure of calculating the circumference of the circle, using square roots, the *Yuktibhāṣā* now points out that it is possible to avoid the cumbersome computation of square roots, and proceeds to calculate the circumference using a series expansion. (This is closely analogous to the avoidance of square-root extraction while computing sine values.) Unlike the geometric technique of computing circumference which is restricted to the calculation of π, the infinitesimal techniques can be used also to calculate various trigonometric values. This provides an important link between the computation of the circumference ("π") and the computation of sine values proper, using Āryabhaṭa's finite