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2.5 Preacceleration: the Taylor-series approximation

We saw in Chapter VA that the study of radiative damping and, in particular, the Schott term, leads to equations that are of the third order in time, resulting in the preacceleration of the electron. Dirac, in 1938, obtained these equations by means of a Taylor expansion which seems unavoidable.⁴ Many other authors⁵ have attempted similar approximation procedures, using a Taylor series expansion to get rid of retarded/advanced expressions, in dealing with the two-body problem in electrodynamics and gravitation. Physically, this procedure means that we model a history-dependent system by an instantaneous system with additional degrees of freedom.

This procedure is known to be, in general, invalid. This may be seen from the following counter-example,

$$x'(t) = -2x(t) + x(t-r), \quad (18)$$

where $r > 0$ is a small constant. Every solution of this equation is bounded⁶ and tends to zero as $t \rightarrow \infty$. But if we choose the Taylor series approximation to the right hand side and truncate after two terms, we obtain

$$x'(t) = -2x(t) + [x(t) - rx'(t) + \frac{1}{2}r^2x''(t)], \quad (19)$$

which admits exponentially increasing solutions $x(t) = c \exp(\alpha t)$, with $\alpha > 0$. Thus, the Taylor approximation of (18) by (19) leads to qualitatively incorrect behaviour, no matter how small r is, so long as $r > 0$.

It may be shown that it is not the order of the approximation which is at fault: with instantaneous data, even an infinite number of degrees of freedom is inadequate. The order of the approximation does, however, make a difference from the numerical point of view, as pointed out by El'sgol'ts,⁷ 'since the transition is equivalent to the rejection of the term with the highest order derivative in an *unstable-type differential equation* with a small coefficient before the highest derivative.' [Emphasis mine]

In the usual treatment of the numerical solution of retarded o.d.e.'s, attention is focused upon the discontinuities that might arise at the ends of delay intervals (e.g. Solution 1 of Fig. 1). However, one would expect the general electrodynamic many-body problem to be 'stiff': there could be oscillations at widely varying frequencies. In view of the Dahlquist barrier,⁸ A-stability fails for any rule higher than the trapezoidal rule, so that the Taylor approximation could be numerically misleading for derivatives of order greater than two.⁹ Thus, Dirac was perhaps right in a way when he rejected the higher order terms as too complex to apply to 'a simple thing like the electron'.

To summarize, the origin of the Schott term in the Lorentz-Dirac equation of motion is mathematically dubious, and can result in qualitatively incorrect behaviour, though it may yet provide a more robust numerical approximation than would be obtained by the inclusion of higher-order relativistically covariant terms. The alternatives that have been proposed,¹⁰ to the Lorentz-Dirac equation, have not proved satisfactory.¹¹

5. L. Page, Phys. Rev., **12**, 371 (1918); **24**, 296 (1924); A. Einstein, L. Infeld and B. Hoffmann, Ann. Math., **39**, 65 (1938); H. P. Robertson, Ann. Math., **39**, 101 (1938); L. Infeld, Phys. Rev., **53**, 836 (1938); A. Eddington and G. L. Clark, Proc. R. Soc., **A166**, 465(1938); A. Einstein and L. Infeld, Can. J. Math., **1**, 209(1949); G. L. Clark, Proc. R. Soc. (Edinb.), **A64**, 49 (1954); B. Bertotti, Nuovo Cim., **12**, 226 (1954). A more detailed list may be found in the article by P. Havas in *Statistical Mechanics of Equilibrium and Non-Equilibrium*, ed. J. Meixner (Amsterdam: North Holland, 1965). More recently, somewhat similar approximations have been attempted by L. P. Grishchuk and S.M. Kopejkin, in: J. Kovalevsky and V.A. Brumberg (eds), *Relativity in Celestial Mechanics and Astronomy*, IAU (1986) pp 19-34; V.I. Zhdanov, J. Phys. A: Math. Gen., **24**, 5011-27 (1991).