Calculus without Limits

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Extended synopsis

Calculus is today taught[1] as a bunch of rules. Students learn $(e^x)' = e^x$ without learning either a good definition of e^x (which requires a study of power series, uniform convergence etc.) or how to calculate it. Understanding is restricted to the primitive geometrical interpretation of derivative and integral.

It is believed that limits must be taught to bring in rigor and clarity. (Consequently, the 12th std. NCERT maths text has a chapter on limits.) However, limits cannot be taught clearly at this level. Thus, limits presuppose at least an account of the formal real numbers $\mathbb{R}$. And *that* requires Dedekind cuts, or equivalence classes of Cauchy sequences, which are not taught. Therefore, limits are taught in a naive way, and the serious part is typically postponed to a course on advanced calculus[2] or topology and mathematical analysis.[3] Although such an advanced course typically starts off by constructing the formal real numbers $\mathbb{R}$, it would still typically omit the requisite *axiomatic* set theory,[4] as distinct from *naive* set theory[5]). Indeed, most school and college texts on calculus start off by saying "a set is a collection of objects", leaving the student befuddled and in a state of perpetual confusion. (Indeed, even few mathematicians ever learn the *definition* of a set.) On the other hand, mere symbolic manipulation skills without understanding have become largely irrelevant over the last two decades, with the easy availability of symbolic manipulation programs like MACSYMA (now open source) or MATHEMATICA which any child can use to solve, in a fraction of a second, the toughest symbolic manipulation problems in any calculus text. (Demo available on request.) Thus, after a calculus course, the student neither acquires conceptual clarity, nor learns anything of significant practical value.

This situation arises from the belief that limits are necessary for the calculus. But why at all are limits necessary? There is a massive underlying irony here. As I have shown,[6] the calculus actually developed in India over a thousand year period, starting from Āryabhaṭa (5th c.), and culminating in Madhava's (15th c.) table of trigonometric values precise to the 9th decimal place. These

[1] e.g. H. Flanders, R. Korfhage and J. Price, *Calculus*, Academic Press, New York, 1970.

[2] e.g. D. V. Widder, *Advanced Calculus,* 2nd ed., Prentice Hall, New Delhi, 1999.

[3] e.g. W. Rudin, *Principles of Mathematical Analysis*, McGraw Hill, New York, 1964.

[4] e.g. L. Mendelson, *Introduction to Mathematical Logic,* van Nostrand Reinhold, New York, 1964.

[5] e.g. P. R. Halmos, *Naive Set Theory*, East-West Press, New Delhi, 1972.

[6] C. K. Raju, *Cultural Foundations of Mathematics: The Nature of Mathematical Proof and the Transmission of the Calculus from India to Europe in the 16th c. CE*, Pearson Longman, 2007, PHISPC vol X.4.

works of the Āryabhaṭa school in Kerala were translated from the Malayalam by Jesuits in their Cochin college in the 16th c., with the help of locals, and transmitted to Europe which then badly needed accurate trigonometric values for their biggest scientific challenge of the times: navigation. Trigonometric values were used for the construction of the Mercator chart (loxodromes), and determination of latitude and longitude at sea. (As pointed out by Struik[7] the calculation of loxodromes is equivalent to the fundamental theorem of calculus.)

However, the initial recipients of these Indian texts (Christoph Clavius, Tycho Brahe, Johannes Kepler) had a very fragmentary understanding (thus Clavius published trigonometric tables to the 9th decimal place, but did not know how to calculate the radius of the earth, etc.). Obviously, in the atmosphere of the Inquisition, no one dared publicly acknowledge their non-Christian sources, least of all these high religious officials, or people like Mercator, arrested by the Inquisition. Galileo, whose access to the Jesuit Collegio Romano is well documented, objected to the (Indian) methodology and passed on the credit (or discredit) for the calculus to his student Cavalieri. Similar objections were raised by René Descartes to the enthusiastic use of these techniques by Fermat and Pascal. Indeed, Descartes went so far as to say in his *Geometry* that understanding the ratio of curved and straight lines was beyond the capacity of the human mind.[8] (Descartes misunderstood the Indian calculus techniques as requiring supertasks—an infinite series of tasks; this misunderstanding is the basis of the belief that limits are required for the calculus.) It was these difficulties that Newton thought he had addressed with his theory of fluxions, although Berkeley later showed him to be wrong. Eventually, these difficulties were addressed through limits, formal real numbers, and the axiomatisation of set theory (which permits supertasks to be performed metaphysically).

Thus limits came to be regarded as necessary for the calculus only because Europeans misunderstood the Indian techniques of calculus, and, although the calculus developed in India, today we teach calculus by re-importing and copying its subsequent evolutionary trajectory in Europe, based on this misunderstanding. As underlined by my "epistemic test", failure to understand is the hallmark of transmission, as also happened when this author's work was repeatedly credited to a Field medallist.[9]

According to my analysis, the European difficulties with the Indian calculus arose because *religious* beliefs are inseparably intertwined with Western mathematical philosophy[10] The remedy,

[7] D. J. Struik, *A Source Book on Mathematics, 1200–1800*, Harvard University Press, Cambridge, Mass., 1969, p. 253.

[8] R. Descartes, *The Geometry* (trans. David Eugene and Marcia L. Latham), Encyclopaedia Britannica, Chicago, 1996, book 2, p. 544.

[9] In my book *Time: Towards a Consistent Theory* (Kluwer Academic, Dordrecht, 1994), I had argued that physics must be done using functional differential equations. This idea was repeated by Michael Atiyah in his 2005 lectures, claiming it was his own suggestion. Although Atiyah was immediately informed of my previous work, my work was *again* overlooked, a second time, and my idea was named as "Atiyah's hypothesis" in an article by G. W. Johnson and M. E. Walker, published in the *Notices of the AMS* in 2006, which was written in consultation with Atiyah, and emphasized that the idea amounted to a potential paradigm shift. My work from 1994–2004 was only belatedly acknowledged a full year later (M. E. Walker, "Retarded Differential Equations and Quantum Mechanics", *Notices of the American Mathematical Society* **54**(4) 2007, p. 473). However, the *Notices* failed to carry my rejoinder, pointing out *Atiyah's mistake*, despite a "petition against celebrity justice" signed in my support by prominent academicians such as M. G. K. Menon, Pushpa Bhargava, Vandana Shiva, A. N. Mitra, and a host of others. For more details and relevant documents see `http://ckraju.net/atiyah/petition/Atiyah-press-release.html`

[10] For a quick account, see C. K. Raju, "The Religious Roots of Mathematics", *Theory, Culture & Society* **23**(1–2) Jan-March 2006, Spl. Issue ed. Mike Featherstone, Couze Venn, Ryan Bishop, and John Phillips, pp. 95–97. For an account of the way in which this "theology of reason" is applied to current politics, see C. K. Raju, "Benedict's Maledicts", http://zmag.org/znet/viewArticle/3109 or the printed version in *Indian Journal of Secularism* **10**(3)

therefore, is to de-theologise mathematics, and to teach it in the original practical context in which it developed.[11]— This would also help to make mathematics both easy and secular by focussing it on practical value, rather than obscure theological concerns.

However, as a former British colony, we lack the academic freedom to make any decision which deviates from the education system in the West, and few such decisions have been made in over 60 years since independence, even though it would be a matter of great practical advantage to teach calculus in the new way, by going back to its old roots in India as a numerical technique for solving ordinary differential equations, since Āryabhaṭa.[12]

Nevertheless, in anticipation of the day when academics in India will have the freedom to educate students for practical advantage, rather than teach them only to rigidly mimic the West, I have suggested a course on the "calculus without limits", taught as the study of the numerical solution of differential equations. This approach is **completely rigorous** (on the *new* philosophy of mathematics I have proposed[13] now called "zeroism", which is contrary to formalism, and achieves all the goals of the calculus *without* invoking supertasks or limits). It also makes calculus shockingly simple. For example the exponential function is defined as the solution of $y'(x) = y(x)$ with $y(0) = 1$. This also makes it very easy to teach "advanced" topics, such as elliptic integrals, even to school children.[14] The students can either write the required computer programs themselves, or use my software CALCODE, which accepts symbolic input to define a differential equation, and provides both numerical output and 3-D visualisation of solutions using Open-GL. CALCODE also has features to analyse the solutions in various ways. (They can even do the calculations by hand.)

As a side benefit, students will learn more than is taught in current calculus courses, so that they become proficient in the sort of elementary physics that is today declared as "advanced", but is necessary[15] to teach the "scientific method": for example, the non-isochrony of the simple pendulum,[16] or why a heavier cricket ball can be thrown further than a lighter tennis ball and by how much.

The only disadvantage of this course is that it represents a departure from mimesis of the West, which has long been the key objective of our education system.

(2008) pp. 79–90.

[11] C. K. Raju, "Math Wars and the Epistemic Divide in Mathematics", Paper presented at Episteme1, Goa, 2004, http://www.hbcse.tifr.res.in/episteme1/themes/ckraju_finalpaper.

[12] C. K. Raju, *Cultural Foundations of Mathematics: the Nature of Mathematical Proof and the Transmission of the Calculus from India to Europe in the 16th c. CE*, Pearson Longman, 2007 (PHISPC vol X.4).

[13] C. K. Raju, *Cultural Foundations of Mathematics*, cited above.

[14] For an actual project along these lines, see `http://ckraju.net/11picsoftime/pendulum.pdf`.

[15] C. K. Raju, "Time: What is it That it Can be Measured?" *Science & Education* **15**(6)(2006) pp. 537-551. Draft available at http://ckraju.net/papers/ckr_pendu_1_paper.pdf.

[16] Ibid.