PROBABILITY IN ANCIENT INDIA

C. K. Raju

INTRODUCTION: MATHEMATICAL PRE-REQUISITES

Permutations and Combinations

A first course on probability (at the high-school level) typically begins with an account of the theory of permutations and combinations needed for calculating probabilities in games of chance, such as dice or cards.

Like many other aspects of mathematics, this theory of permutations and combinations first developed in India, although an account of its history is usually missing in stock presentations of combinatorics.

In fact, the theory of permutations and combinations was basic to the Indian understanding of metre and music. The Vedic and post-Vedic composers depended on combinations of two syllables called *guru* (deep, long) and *laghu* (short). The earliest written account of this theory of metre is in Piñgala's *Chandahsūtra* (-3^{rd} c. CE), a book of aphorisms (*sūtra*-s) on the theory of metre (*chanda*). To calculate all possible combinations of these two syllables in a metre containing n syllables, Piñgala gives the following rule[1] (which explicitly makes use of the symbol for zero). "(Place) two when halved;" "when unity is subtracted then (place) zero;" "multiply by two when zero;" "square when halved." In a worked example, Dutta and Singh[2] show how for the Gāyatrī metre with 6 syllables this rule leads to the correct figure of 2^6 possibilities.

That this rule basically involves the binomial expansion is made clear by Piñgala's commentator the 10^{th} c. CE Halayudha. Thus, in a 3-syllabic metre with two underlying syllables, *guru* and *laghu*, 3 *guru* sounds will occur once, 2 *gurus* and 1 *laghu* will occur twice, as will 1 *guru* and 2 *laghus*, while 3 *laghus* will occur once. Symbolically $(g + l)^3 = g^3 + 3g^2l + 3gl^2 + l^3$. To generalize this to the case of n underlying syllables, Halayudha explains the *meru-prastāra* (pyramidal expansion) scheme for calculation,[3] which is identical to "Pascal's" triangle which first appeared in Europe about a century before Pascal (on the title page of the

[1] *Chandahsūtra* viii.28. Ed. Sri Sitanath, Calcutta, 1840. Cited by Dutta and Singh, vol. 1, p. 76.

[2] B. B. Dutta and A. N. Singh, *History of Hindu Mathematics*, Asia Publishing House, 1962, vol. 1, pp. 75–76.

[3] A. K. Bag, *Mathematics in Ancient and Medieval India*, Chaukhambha Orientalia, Varanasi, 1979, pp. 189–93.

Handbook of the Philosophy of Science. Volume 7: Philosophy of Statistics.
Volume editors: Prasanta S. Bandyopadhyay and Malcolm R. Forster. General Editors: Dov M. Gabbay, Paul Thagard and John Woods.

Arithmetic of Apianus) and in China in the 14^{th} c.[4] An example, using the *Gāyatrī* meter is also found in Bhaskara's *Līlāvatī*.[5] The accounts found in stock Western histories of mathematics (such as that by Smith[6]) incorrectly state that no attention was paid in India to the theory of permutations and combinations before Bhaskara II (12^{th} c. CE).

Although this theory is built into the Vedic metre, the earliest known written account relating to permutations and combinations actually comes from even before Piñgala, and is found in the -4^{th} c. Jain *Bhagwatī Sūtra*. Permutations were called *vikalpa-gaṇita* (the calculus of alternatives), and combinations *bhanga*. The text works out the number of combinations of n categories taken 2, 3 etc. at a time.

Incidentally, this throws up large numbers of the sort that cannot easily be written in Greek (Attic) and Roman numerals. It should be noted that while the Yajurveda[7] already used a place value system, and gives names for numbers up to 10^{12}, the Jain literature typically runs into very large numbers, such as 10^{60}, and the Buddha when challenged (perhaps by a Jain opponent) names numbers up to 10^{53}. Large numbers have an intimate connection with the philosophy of probability, as examined in more detail, later on.

From the earliest Vedic tradition, there is a continuous tradition linking the first accounts of permutations and combinations with those of Bhaskara II (12^{th} c.), and later commentaries on his work, up to the 16^{th} c. CE, such as the *Kriyākramkarī*.[8] Thus,[9] the surgeon Suśruta (-2^{nd} c. CE) in his compendium (*Suśruta-samhitā*) lists the total number of flavours derived from 6 flavours taken 1 at a time, 2 at a time, and so on. Likewise, Varāhamihira (6^{th} c.) who reputedly wrote the first Indian text on astrology (*Bṛhat-Jātaka*) states in it the number of perfumes that can be made from 16 substances mixed in 1, 2, 3, and 4 proportions.

Similar examples are found in the *Pāṭīgaṇita* (Slate Arithmetic) of Śridhar (10^{th} c.), a widely used elementary school-text, as its name suggests, Mahavira's (8^{th} c.)

[4] Joseph Needham, *The Shorter Science and Civilisation in China*, vol. 2 (abridgement by C. A. Ronan). Cambridge University Press, 1981, p. 55.

[5] Bhaskara, *Līlāvatī*, trans. K. S. Patwardhan, S. A. Naimpally, and S. L. Singh, p. 102. The verse is numbered differently in different manuscripts. K. V. Sarma in his critical edition of the 16^{th} c. southern commentary *Kriyākramakarī* (VVRI, Hoshiarpur, 1975) on the *Līlāvatī*, gives this as verse number 133, while the other cited source has given it as verse number 121.

[6] D. E. Smith, *History of Mathematics*, Dover Publications, 1958, vol. 2, p. 502.

[7] Yajurveda xvii.2 gives the names for the first 12 powers of 10, the first five being more or less similar to what they are today.

[8] This has an interesting connection with the history of the calculus. Fermat's challenge problem is identical with a solved exercise in Bhaskara II. So this is one of the texts that travelled from Cochin to Rome, and Bhaskara was probably Pascal's source. *Bījagaṇita of Srī Bhāskarācārya*, ed. Sudhakara Dvivedi, Benares, 1927 (Benares Sanskrit Series, No. 159), chapter on *cakravāla*, p. 40. An account of Bhaskara's *cakravāla* method may be found, for instance, in Bag, cited above (pp. 217–228). For a formalised account of Bhaskara's *cakravāla* method, see I. S. Bhanu Murthy, *A Modern Introduction to Ancient Indian Mathematics*, Wiley Eastern, New Delhi, 1992, pp. 114–21. (Bhanu Murthy's book has a typo here.) For Fermat's challenge problem and "Pell's equation", see D. Struik, *A Source Book in Mathematics 1200–1800,* Harvard University Press, Cambridge, Mass., 1969, pp. 29–30.

[9] A. Bag, *Mathematics in Ancient and Medieval India*, cited above, p. 188.

Gaṇita Sāra Saṁgraha, and Bhaskara II (*Līlāvatī*) etc. Bhaskara mentions that this formula has applications to the theory of metre, to architecture, medicine, and *khaṇḍameru* ("Pascal's triangle"). In these later texts, one finds explicitly stated formulae for permutations and combinations.

For example, to calculate $\binom{n}{r}$ values, Śridhar, in his text on slate-arithmetic[10] (*Pāṭigaṇita*), gives the following rule.

एकाद्युत्तरविधिना रस विन्यासे विलोमतो गुण्येत्
पूर्वेण परं क्रमशो रूपादिचयैर्हरैर्विभ्जेत्

This translates as follows (*Pāṭigaṇita*, 72, Eng. p. 58)

> Writing down the numbers beginning with 1 and increasing by 1 up to the (given) number of savours in the inverse order, divide them by the numbers beginning with 1 and increasing by 1 in the regular order, and then multiply successively by the preceding (quotient) the succeeding one. (This will give the number of combinations of the savours taken $1, 2, 3, \ldots$, all at a time respectively.)[11]

Thus, in the case of 6 savours, one writes down the numbers 1 to 6 in reverse order

$$6, 5, 4, 3, 2, 1$$

These are divided by the numbers in the usual order, to get the quotients

$$\frac{6}{1}, \frac{5}{2}, \frac{4}{3}, \frac{3}{4}, \frac{2}{5}, \frac{1}{6}.$$

Then, according to the rule, the number of combinations of savours taken 1 at a time, 2 at a time, etc., up to all at a time are respectively

$$\frac{6}{1}, \frac{6}{1}\times\frac{5}{2}, \frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}, \text{ etc.}$$

Although, the formulae are mostly stated in identical terms, they are applied most flamboyantly by Bhaskara II. For example, to illustrate one of his formulae, Bhaskara asks for the total number of 5 digit numbers whose digits sum to 13.[12]

पंचस्थानस्थितैरंकैर्यद्यद्योगस्त्रयोदश
कतिभेदा भवेत्संख्या यदि वेत्सि निगद्यताम्

He then adds in the next verse that although this question involves "no multiplication or division, no squaring or cubing, it is sure to humble the egotistical and evil lads of astronomers".

---

[10] Śridhar, *Pāṭigaṇita*, 72, ed. & trans. K. S. Shukla, Dept. of Mathematics and Astronomy, Lucknow University, 1959, Sanskrit, p. 97.

[11] *Pāṭigaṇita of Śridhar*, trans. K. S. Shukla. As he points out, similar articulations are found in the *Gaṇita Sāra Saṃgrah* of Mahavira, vi.218, *MahāSiddhānta* of Āryabhaṭa 2, xv, 45–46 etc.

[12] *Līlāvatī of Bhāskarācārya*, trans. Patwardhan et al., p. 181. They give the number of this verse as 276, whereas, in K. V. Sarma's critical edition of *Kriyākramkarī*, a commentary on the *Līlāvatī*, this is at 269, p. 464.

Weighted averages

The notion of simple average was routinely used in Indian planetary models, where each planet had a mean motion, and a deviation from it. Unlike Western planetary models, there was no belief in any divine harmony nor any faith in divine "laws" of any sort involved here, just an average motion and deviations from it in a down-to-earth empirical sense. While the deviations from the mean were not regarded as necessarily *mechanically* explicable, neither were they regarded as quite "random", for it was believed that the deviation could be calculated in principle, at least to a good degree of approximation (required for the Indian calendar, which identified the rainy season, and hence was a critical input for monsoon-driven agriculture in India).

In various elementary mathematical texts, in the context of computing the density of mixtures or alloys, one also finds the usual formula for weighted averages, which is so closely related to the notion of "mathematical expectation" in probability theory. For example, we find in verse 52(ii) of the *Pāṭīgaṇita* (Eng. p. 36) the following:

> "The sum of the products of weight and *varṇa* of the several pieces of gold, being divided by the sum of the weights of the pieces of gold, give the *varṇa* (of the alloy).

That is, if there are n pieces of gold of weights $w_1.w_2, \ldots, w_n$, and *varṇas* $v_1, v_2, \ldots, v_n$, the *varṇa* v of the alloy is given by $v = \frac{w_1v_1+w_2v_2+\cdots+w_nv_n}{w_1+w_2+\cdots+w_n}$.

(The term *varṇa* is analogous to the term "carat", with pure gold consisting of 16 *varṇas.*) Understandably, this topic of "mixtures" is given special emphasis in Jain texts like those of Mahavira.

The relevance of these weighted averages to gambling was understood. It is in this very context of "mixtures" that the *Gaṇita Sāra Saṃgraha*[13](268.5, 273.5) gives the example of "Dutch bets" mentioned by Hacking.[14] This is "a rule to ensure profit (in gambling) regardless of victory or loss", a method of riskless arbitrage, in short. The text illustrates the rule with an example, where "a great man knowing mantra and medicine sees a cockfight in progress. He talks to the owners of the birds separately in a mysterious way. He tells one that 'if your bird wins, you give me the amount you bet, and if it loses, I will give you $\frac{2}{3}$ of that amount'. Then he goes to the owner of the other bird where on those same conditions he promised to pay $\frac{3}{4}$ of the amount. In either case, he earned a profit of only 12 pieces of gold. O mathematician, blessed with speech, tell me how much money did the owner of each bird bet."

[13] *Gaṇitasāra-Saṃgraha*, (Hindi trans. L. C. Jain), Jain Samskrti Samraksha Sangh, Sholapur, 1963, pp. 159–60.

[14] Ian Hacking, *The Emergence of Probability: A Philosophical Study of Early Ideas about Probability, Induction and Statistical Inference*, Cambridge University Press, 1975, pp. 6–9. Hacking uses the English translation of *Gaṇita Sāra Saṃgraha*, trans. M. Rangacharya (1912), pp. 162–3.

Precise fractions

Apart from the ability to work with large numbers, and to calculate permutations and combinations, and weighted averages, there is also needed the ability to work with fractions having large numerators and denominators. Such ability, indicative of greater precision, is not automatic. Such precise fractions with large numerators and denominators are certainly found in Indian mathematical texts from the time of Āryabhaṭa.[15] Their use is also reflected in the Indian calendar.

By way of comparison, the Romans had only a few stock fractions to base 12 (each of which had a separate name). Hence they had a wrong duration of the year as $365\frac{1}{4}$ days, just because it involved such an easy-to-state fraction. And they retained this wrong duration even after the repeated calendar reforms of the 4th to 6th c., and, for over a thousand years, until 1582. The Greek Attic numerals were similar, and the Greeks did not work with such fractions in any text coming actually (and not notionally) from before the 9th c. Baghdad House of Wisdom.[16] The significance of the Baghdad House of Wisdom is that Indian arithmetic texts travelled there, were translated into Arabic, and some of these were further translated from Arabic into Greek[17] like the Indian story book, the *Pañcatantra.*[18] (All known Arabic and Greek manuscripts of "Ptolemy's" *Almagest,* for example, post-date the Baghdad House of Wisdom—most also post-date the Crusades—and are decidedly post-9th c. accretive texts as is clear, for example, from their star lists. Therefore, it would be anachronistic to attribute, uncritically and ahistori-

[15] *Āryabhaṭiya*, *Gītikā* 3-4, trans. K. S. Shukla and K. V. Sarma, INSA, New Delhi, 1976, p. 6. This verse gives revolution numbers for various planets, and requires us to calculate fractions such as 1582237500/4320000. Up to a hundred years ago, Western historians, who subscribed to the view that the world was created a mere 6000 years ago, invariably described these figures as fantastic cosmological speculations, and meaningless by implication. However, the above fraction leads to a surprisingly accurate figure of 365.25868 days for the length of the *sidereal* year (compared to its modern value of 365.25636 days). In contrast, the value attributed to Ptolemy (365.24666 days) is significantly less accurate, and we know that the length of the (tropical) year on the Roman calendar was less accurate than the figure attributed to "Ptolemy". This led to a slip in the date of Easter, and hence necessitated the repeated attempts at Roman calendar reform in the 5th and 6th centuries.

[16] The difference between actual and notional dates is important. The only clear way to check a notional date is through the non-textual evidence, and it is hard to believe that the crude Greek and Roman calendars, despite repeated attempts by the state and church to reform them, could have co-existed for centuries with relatively sophisticated astronomy texts, such as the *Almagest* attributed to Ptolemy (which attribution makes its notional date 2nd c., although the actual manuscripts of it come from a thousand years later, and are accretive).

[17] C. K. Raju, *Cultural Foundations of Mathematics*, Pearson Longman, 2007. This text examines in detail the points made in note 16 above. Also, C. K. Raju, *Is Science Western in Origin?*, Citizens International and Multiversity, Penang, 2009, contains a brief account.

[18] Edward Gibbon, *The Decline and Fall of the Roman Empire*, Great Books of the Western World, vols 37–38, Encyclopaedia Britannica, Chicago, 1996, vol. 2, note 55 to chp. 52, p. 608. Others have assigned the date of 1080 to Simon Seth's Greek translation of the *Pañcatantra* from Arabic. The Arabic translation *Kalilah va Dimnah* by Ibn al Muqaffa (d. 750), was long before the formation of the House of Wisdom, and the movement called the Brethren of Purity (Ikhwan al-Safa) derives inspiration from this text. This translation was from the Pahlavi translation which was from the Sanskrit, and done by Burzoe himself, the vazir of Khusrow I, Noshirvan, according to the *Shahnama* of Firdausi.

cally, the use of sexagesimal fractions in such a late text to a mythical Claudius Ptolemy of the 2nd c.[19])

The game of dice in India

Thus, Indian tradition had all the arithmetic tools needed for the calculation of discrete probabilities. But was there a concept of probability? Stock mathematics texts draw their examples from a range of sources, and we also occasionally find some examples related to games of chance such as dice. The same slate-arithmetic text (*Pāṭīgaṇita*, 99–101, p. 145) gives a long and complex rule for calculating whether one has won or lost in a game of dice. However, this is given as an application of the formulae for arithmetic progression. The substance of these formulae is explained as follows.

Suppose that two persons A and B gamble with dice, and that they alternately win $p_1, p_2 p_3 p_4$ casts. If the stake-moneys of the casts be in the arithmetic progression $a, a+d, a+2d, \ldots$ then the amount won by $A = [a + (a+d) + (a+2d) + \ldots (p_1 \ terms)] + [a + (p_1+p_2)d + a(p_1+p_2+1)d + \ldots (p_3 \ terms)]$. Likewise, the amount won by $B = [a + p_1 d + a + (p_1+1)d + \ldots (p_2 \ terms)] + [a + (p_1+p_2+p_3)d + +(p_1+p_2+p_3+1)d + \ldots (p_4 \ terms)]$ and this leads to the enunciated rule.

This suggests that the game of dice might not have been played the same way in India, as it is played today, and also that a common strategy followed was (somewhat like martingale bets) to go on increasing the stake as the game went on. But this still does not give us enough information. (Accounts of the game of dice are *not* found in the *Gaṇita Sāra Saṃgraha*.[20] Presumably, the Jains did not want their children to start thinking about such things!) The text naturally takes it for granted that the readers are familiar with this game of dice, but *we* do not seem to have adequate sources for that at the moment, or at least such sources are not known to this author.

The hymn on dice in the Ṛgveda

Games of chance, such as dice, certainly existed in Indian tradition from the earliest times. We find an extraordinary *akṣa sūkta* or hymn on dice in the Ṛgveda (10.2.34). The long hymn begins by comparing the pleasure of gambling with the pleasure of drinking soma!

"There is enjoyment like the soma in those dice".[21] (सोमस्येव मौजवतस्यमचो)

It goes on to describe how everyone avoids a gambler, like an old man avoids horses, even his mother and father feign not to recognize him, and he is separated from his loving wife. Many times the gambler resolves to stay away, but each time the fatal attraction of the dice pulls him back. With great enthusiasm he

[19]This is argued in more detail in C. K. Raju, *Cultural Foundations of Mathematics*, cited above. See also notes 15 and 16 above.

[20]Mahavir, *Gaṇita Sāra Saṃgraha*, Hindi trans. L. C. Jain, Jain Samskrti Samrakshaka Sangha, Sholapur, 1963.

[21]Ṛgveda, 10.2.34.1

reaches the gambling place, hoping to win, but sometimes he wins and sometimes his opponent does. The dice do not obey the wishes of the gambler, they revolt. They pierce the heart of the gambler, as easily as an arrow or a knife cuts through the skin, and they goad him on like the ankus, and pierce him like hot irons. When he wins he is as happy as if a son is born, and when he loses, he is as if dead. The 53 dice dance like the sun playing with its rays, they cannot be controlled by the bravest of the brave, and even the king bows before them. They have no hands, but they rise and fall, and men with hands lose to them. The gambler's wife remains frustrated and his son becomes a vagabond. He always spends his night in other places. Anyone who lends him money doubts that he will get it back. The gambler who arrives in the morning on a steed leaves at night without clothes on his back. [Such is the power of dice!] O Dice, I join the ten fingers of my hands and bow to the leader among you!

THE NOTION OF A FAIR GAME AND THE FREQUENTIST INTERPRETATION OF PROBABILITY

Fair and deceitful gambling in the Mahabharata

This tells us a great deal about the social consequences of gambling, but still very little about how exactly the game was played. (This is naturally assumed to be known to all.) But we do find something interesting from a philosophical perspective on probability. For, from the earliest times, there was also a notion of what constitutes a "fair game", a notion which is today inextricably linked to the notion of probability.

This can be illustrated by a story from the Mahabharata epic. A key part of the story, and the origin of the Mahabharata war, relates to the way the heroes (Pandava-s) are robbed of their kingdom by means of a game of dice (द्युत क्रीडा). They cannot very well refuse the invitation to play dice because the game involves risk, and a Kshatriya is dishonoured by refusing to partake in an enterprise involving risk. However, at the start of the game, Yudhisthira, the leader of the Pandavas, makes clear that he knows the dice are loaded, or that the game will involve deceit. He says: "निक्रितिर्देवनं पापं न चात्रोत्र पराक्रमः" (Mbh, Sabhā Parva, 59.5). That is, "Deceitful gambling is sinful, there is no Kshatriya valor in it."

Philosophically speaking, there clearly are two concepts here: "deceitful gambling" as opposed to "fair gambling". The notion of a fair game, though not explicitly defined, is critical to the story, and involves some notion of probability implicit or explicit. The Mahabharata narrative itself brings out the unfairness of the game both through Yudhishthira's statement, at the beginning of the game, and by telling us about a long series of throws in each one of which Yudhishthira loses, without ever winning once. Of course, there is room to argue, as the devil's advocate might, and as Duryodhana's uncle Shakuni does, that it is a case of gambler's ruin or a long run of bad luck with a finite amount of capital howsoever large. Apart from the Yudhisthira-Shakuni dialogue, the Mahabharata narrative itself carries the depiction of this unfairness to the utmost. Yudhisthira goes on losing, and in desperation, he even stakes his wife, Draupadi, who is then deemed to have been won, and publicly stripped and insulted by Duryodhana.[22]

Interestingly, a little later in the same epic, Yudhisthira, now banished to the forest, recounts his woes: how he was cheated out of a kingdom and his wife insulted. To console him, a sage recounts to him the celebrated love story of Nala and Damayanti. King Nala too lost his kingdom—once again through deceitful gambling. Having lost also his last remaining clothes in the forest, he covers himself with half of Damayanti's sari, and abandons her in the forest. He takes up a job as a charioteer with king Ṛtuparṇa of Ayodhya. His aim is to learn the secret of dice from Ṛtuparṇa , for he has been advised by a Naga prince that that knowledge will help him win back his kingdom. He gets an opportunity, at a tense moment, while rushing from Ayodhya to Vidarbha for the announced second marriage (*swayamvara*) of Damayanti whom Ṛtuparṇa too wants to wed. Nala stops near a Vibhı̈ṭaka tree[23]—it is significant that the nuts of this very tree, with five faces, were used in the ancient Indian game of dice—and Ṛtuparṇa can't resist showing off his knowledge of mathematics (*gaṇita vidyā*) by saying: the number of fruits in the two branches of the tree is 2095, count them if you like.

Nala decides to stop and count them. Ṛtuparṇa, who is apprehensive of being delayed and knows that he cannot reach in time without Nala's charioteering skills, suggests that Nala should be satisfied with counting a portion of one branch.

> Then the king reluctantly told him, 'Count. And on counting the leaves and fruits of a portion of this branch, thou wilt be satisfied of the truth of my assertion.'[24]

Nala finds the estimate accurate and wants to know how it was done, offering to exchange this knowledge with his knowledge of horses. "And king Ṛtuparṇa, having regard to the importance of the act that depended upon Vahuka's [Nala's] good-will, and tempted also by the horse-lore (that his charioteer possessed), said, 'So be it.' As solicited by thee, receive this science of dice from me..." (*ibid*). (The story has a happy ending, and Nala does get back his wife and his kingdom.)

From our immediate point of view, the interesting thing is the way the Mahabharata text conflates sampling theory with the "science of dice". (It is legitimate to call this "science", for there is a clear relation to a process of empirical verification, by cutting down the tree and counting.) This connection of sampling theory to the game of dice is mentioned by Hacking,[25] who attributes it to V. P. Godambe. They have, however, overlooked the other aspect of the story, which is that this knowledge was regarded as secret, and Ṛtuparṇa parts with it only under

[22]Eventually, it is decreed that Yudhisthira incorrectly regarded her as his property to be staked, however, one of the heroes, Bhim, swears that he will break Duryodhana's thigh, on which he seats Draupadi, and also drink the blood of his brother Dusshasana who forcibly dragged Draupadi to the court, both of which promises he fulfills years later in the battlefield.

[23]Mahabharata, *van parva*, 72, trans. K. M. Ganguly, 1883–1896, Book 3, pp. 150–51.

[24]*ibid*, p. 151.

[25]Ian Hacking, *The Emergence of Probability*, cited earlier, pp. 6–9.

extraordinary conditions; therefore, one should not expect to find this knowledge readily in common mathematical texts.

"Fair gambling", the law of large numbers

There are, however, repeated references to "deceitful gambling". So, one can ask: what exactly did "fair gambling" mean? As the notion is understood today, in an ultimate sense, the fairness of the game cannot be established merely by the law of large numbers, for it is well known that the frequentist interpretation of probability fails because relative frequency converges to probability only in a probabilistic sense. Shakuni is right that a long streak of bad luck is always possible (and must almost surely occur in the long run). Therefore, also, the randomness of a set of random numbers can *ultimately* be established only by reference to the process which generated those random numbers and not merely by *post-facto* statistical tests of randomness. Note the emphasis on "ultimate". Given a series of random numbers even a simple χ^2 test could show,[26] for most practical purposes, whether the series is concocted; the question is whether it can settle "all" doubt, for there is always some residual probability that the test gave a wrong result, and this problem of what to do with very small numbers takes us back to the old problem of the law of large numbers. One cannot rule out the occurrence even of probability-zero cases, and the best one can say is that they will "almost surely" not occur.

Law of large numbers and the notion of convergence as a supertask

At this point, it is probably a good idea to understand two key differences between the Indian philosophy of mathematics and the Western philosophy of mathematics. The contemporary notion of a fair game already involves some notion of the law of large numbers, hence a notion of *convergence* in some sense (such as convergence in probability). It does not matter how "weak" this notion of convergence is (i.e. whether L^p convergence or convergence in measure): the point is that, as conceptualised in the Western philosophy of mathematics, *any* notion of convergence involves a supertask—an infinite series of tasks. Such supertasks can *only* be performed *metaphysically*, within set theory; however, barring special cases, a supertask is not something that can be empirically performed (in any finite period of time).

Supertasks and the clash of mathematical epistemologies

Historically speaking, the transmission of the calculus[27] from India to Europe in the 16$^{\text{th}}$ c. led to a clash of epistemologies. The situation is analogous to the

[26] E.g., C. R. Rao, *Statistics and Truth: Putting Chance to Work*, CSIR, New Delhi, 1989.

[27] C. K. Raju, *Cultural Foundations of Mathematics,* Pearson Longman, 2007, chp 9, "Math wars and the epistemological divide in mathematics".

clash of epistemologies which occurred when the Indian arithmetic techniques, called the Algorismus (after al Khwarizmi's latinized name), or "Arabic numerals", were imported into Europe and clashed with the abacus or the native Roman and Greek tradition of arithmetic. With regard to the Indian calculus, Western philosophers did not understand its epistemology, and felt that it involved supertasks. The practical usefulness of the calculus (in calculating trigonometric values, navigational charts etc.) then created the need to justify such supertasks. In the Western philosophy of mathematics, this mistaken belief about supertasks arose because of two reasons, (1) a belief in the perfection of mathematics, and (2) a distrust of the empirical as means of proof.

Thus, for example, reacting to the use of the new-fangled calculus-techniques by Fermat and Pascal, Descartes[28] wrote in his *Geometry* that comparing the lengths of straight and curved lines was "beyond the capacity of the human mind". Galileo's reaction, in his letters to Cavalieri, was similar, and he ultimately left it to his student Cavalieri to take credit or discredit for the calculus.

Descartes' reaction seems idiosyncratic and excessive, because any Indian child knew how to measure the length of a curved line empirically by using a piece of string, or *śulba* or *rajju*, and this was done in India, at least since the days of the *śulba sūtra* [−500 CE]. Perhaps, Descartes thought that the European custom of using a rigid ruler was the only way of doing things. (European navigators certainly had a serious problem for just this reason.) In any case, Descartes proceeded from the metaphysical premise that only the length of a straight line was meaningful. On this metaphysical premise, he thought that the length of a curved line could *only* be understood by *approximating* it by a series of straight-line segments. However large the number of straight line segments one might use, and howsoever fine the resulting approximation might be, it remained "imperfect", hence was not quite mathematics, for mathematics (Descartes believed) is perfect and cannot neglect the smallest quantity.

Descartes was willing to make a concession and allow that infinitesimal quantities could be neglected (a premise shared by many of his contemporaries, including, later Berkeley, in his criticism of Newton, and a process later formalised in non-standard analysis). Therefore, the only situation Descartes was willing to contemplate as acceptable was a situation where the neglected error was infinitesimal. But this required an *infinity* of straight line segments, each of *infinitesimal* length. But that created another difficulty: to compute the length of the curved line, one now had to sum an infinity of infinitesimals. And *this*, thought Descartes, was a supertask only God could perform.[29] (This was in the days before formal mathematics. Descartes thought of summing an infinite series by actually carrying out the sum.)

---

[28] René Descartes, *The Geometry*, Book 2, trans. David Eugene and Marcia L. Latham, Encyclopaedia Britannica, Chicago, 1990, p. 544.

[29] For more details, see C. K. Raju, "The Indian Rope Trick", *Bhartiya Samajik Chintan* **7**(4) (New Series) (2009) pp. 265–269. `http://ckraju.net/papers/MathEducation2RopeTrick.pdf`. Also, *Cultural Foundations of Mathematics,* cited earlier, for more details.

That is, with the European way of thinking about mathematics as "perfect", the problem of neglecting small numbers got converted into the problem of supertasks or convergence. This got related to the difficulty of summing an "infinity of infinitesimals". It is these philosophical problems (supertasks, convergence) which are deemed to have been resolved in the Western tradition of mathematics today, through formal set theory, which enables the supertasks needed for the definition of formal real numbers and limits to be carried out formally, and which are needed for the current notion of a "fair game".

To put matters in another way, the philosophical issue which blocks the frequentist interpretation of probability is the Western belief that mathematics is "perfect", and hence cannot neglect even the smallest quantity, such as 10^{-200}. In practice, of course, this frequentist understanding is commonly used in physics and works wonderfully well with a gas in a box where a "large" number may mean only 10^{23} molecules.

In Western thought, however, what is adequate for physics is inadequate for mathematics, because empirical procedures are suspect and regarded as inferior to metaphysical procedures which are regarded as certain and "ultimate". This attitude is, in fact, at the core of almost all Western philosophy: deduction, a purely metaphysical process, is believed to be surer than induction, which involves an empirical process.

(To simplify the discussion, we are not getting here into another fine issue: the very division of physical and metaphysical in terms of Popper's criterion of refutability already involves inductive processes, and their relative valuation, for logical refutability does not guarantee empirical refutability, which latter requires an inductive process. Popper claimed that he had solved the problem of induction,[30] since probabilities, defined the Kolmogorov way, are not ampliative and are left unaffected by any new experiments. However, Popper overlooked the fact that empirically one can only obtain *estimates* of probabilities, never the probabilities themselves, and estimates of probabilities *may* be ampliative. So, Popper did not succeed in throwing any new light on the problem of induction.)

Suffice to say that these difficulties regarding "the fear of small numbers" and "inferiority of the empirical" are closely related to the understanding of the "law of large numbers" on the one hand, and , on the other hand, to the notion of "convergence" and "limits" that developed when Europeans tried to assimilate the calculus imported from India in the 16$^{\text{th}}$ c. CE.

Zeroism: beyond the clash of epistemologies

The hate politics that prevailed for centuries in Europe during the Crusades and Inquisition had a decisive impact not only on the Western history of science, but also on its philosophy: both had to be theologically correct (on pain of death). The way this was exploited by colonialists and racists is made abundantly clear

[30] K. R. Popper, *Realism and the Aim of Science*, PS to LscD, vol 1, Hutchinson, London, 1982.

in various ways. For example, Frank Thilly's text on philosophy, used as a text in pre-independence India, starts with the dismissal of all non-Western traditions as "non-philosophy". This belittling of anything non-Western as inappropriate and not-quite-science still represents a typical Western attitude today.

However, if we abandon the current Western historical myths about mathematics in Indian tradition, and look at the tradition *per se*, that gives us a fresh angle on the philosophical problems related to small numbers, and supertasks and the alleged unreliability of empirical procedures, which philosophical problems are intimately linked to the notion of convergence, and the related issue of the frequentist interpretation of probability.

First, let us take note of the differences in the cultural context. Basically, in India, unlike Europe, there was never a monopoly or hegemony of any single religion, and in this pluralistic (and largely secular) Indian environment, where many different brands of metaphysics were prevalent, there was no question of treating any particular brand of metaphysics as certain. Different people may (and did) have conflicting beliefs even about logic from the earliest times.[31] The Lokāyata rejected deduction as an inferior technique which did not lead to sure knowledge. The tradition in India, since the time of the Buddha, was to decide truth by debate, and not by the forcible imposition of any one brand of metaphysics.

Since metaphysics carried no certitude, the empirically manifest (*pratyakṣa*) was the primary basis of discourse for two persons with conflicting metaphysical beliefs (even about logic etc.). This was the one (and only) means of proof which was accepted by *all* philosophical schools of thought in India. All this is in striking contrast to Western tradition where mathematics is regarded as metaphysical, and this metaphysics is simply declared, by social fiat, to be "universal" and certain like a religious belief. This attitude emerged naturally in the West, where mathematics was linked to religion and theology, and was spiritual and anti-empirical since the days of Plato and Neoplatonists like Proclus. (Indeed the very name "mathematics" comes from mathesis, meaning learning, and learning, on the well-known Platonic/Socratic doctrine, is recollection of knowledge obtained in past lives, and so relates to stirring the soul. Mathematics was thought of as being especially good for this purpose, since it involved eternal truths which moved the eternal soul by sympathetic magic.) It is this belief—that mathematics contained eternal truths—which led to the belief that mathematics is perfect and hence cannot neglect the smallest quantity.

We have seen that the philosophical issue which blocks the frequentist interpretation of probability is just this Western belief in the perfection of mathematics. A number, howsoever small, such as 10^{-200} cannot be neglected and set equal to zero. On the other hand, Indian tradition allowed this to be done, somewhat in the manner in which rounding is done today, but with a more sophisticated philosophy known as *´nyavāda* which I have called zeroism,[32] so as to emphasize the

[31] B. M. Baruah, *A History of Pre-Buddhistic Indian Philosophy*, Calcutta, 1921; reprint Motilal Banarsidass, Delhi, 1970.

[32] C. K. Raju, "Zeroism and Calculus without Limits", 4^{th} dialogue between

fruitful practical aspects of the philosophy, and avoid sterile controversies about the exact interpretation of Buddhist texts.

Now it so happens that the formula for the sum of an infinite (*anantya*) geometric series was first developed in India (we find this in the *Āryabhaṭiya Bhāṣya* of the 15th-16th c. Nīlakaṇṭha[33]). This naturally involved the question of what it *means* to sum an infinite series. A procedure existed to test the convergence of an infinite series (of both numbers and functions), and I have explained it elsewhere in more detail.[34] This is similar to what is today called the Cauchy criterion of convergence, except for one aspect. Thus, given an infinite series of numbers, Σa_n, and an $\epsilon > 0$, the process checked whether there was an N beyond which $\| \sum_{n=N}^{N+m} a_n \| < \epsilon$. The supertask of actually summing the residual partial sums for *all* numbers m could obviously not be carried out except in some special cases (such as that of the geometric series). However, if $S(k) = \sum_{n=1}^{k} a_n$ denotes the partial sums, the usual process was to check whether the partial sums became "constant", beyond some N.[35] Obviously, the partial sums $S(k)$ would never literally become constant, and when successive terms are added, there would always be *some* change (except when all the terms of the series are zero beyond N, so that the infinite sum reduces to a finite sum). So this "constancy" or "no change" was understood to hold only *up to the given level of precision* (ϵ) in use. That is, the sum of an infinite series was regarded as meaningful if the partial sums $S(k)$ became constant, after a stage, *up to a non-representable* (or discardable) quantity: $\|S(N + m) - S(N)| < \epsilon$, which is just the criterion stated earlier. What exactly constitutes a "non-representable" or "discardable" quantity (ϵ) is context-dependent, decided by the level of precision required, and there need be no "universal" or mechanical rule for it.

Apart from a question of convergence, a key philosophical issue which has gone unnoticed relates to *representability.* The decimal expansion of a real number, such as π, also corresponds to an infinite series. Regardless of the convergence of this series, it can be *written* only up to a given number of terms (corresponding to a given level of precision): even writing down the terms in the infinite decimal expansion is a supertask: this is obvious enough when there is no rule to predict what the successive terms would be. So a real number such as π can *never* be accurately represented; Indian tradition took note of this difficulty from the earliest times, with the *śulba sutra*-s (-500 CE or earlier) using the words[36] सविशेष ("with something left out") or[37] सानित्य (*sa* + *anitya* = "impermanent, inexact"), and

Buddhism and Science, Nalanda, 2008. Draft at `http://ckraju.net/papers/Zeroism-and-calculus-without-limits.pdf`.

[33] *Āryabhaṭīya of Āryabhaṭācārya with the Bhāṣya of Nīlakaṇṭhasomasūtvan*, ed. K. Sambasiva Sastri, University of Kerala, Trivandrum, 1930, reprint 1970, commentary on *Gaṇita* 17, p. 142.

[34] C. K. Raju, *Cultural Foundations of Mathematics*, cited above, chp. 3.

[35] *ibid.* p. 177-78, and e.g., *Kriyākramakarī,* cited above, p. 386.

[36] Baudhāyana *śulba sūtra*, 2.12. S. N. Sen and A. K. Bag, *The Śulbasūtras of Baudhāyana, Apastamba, Kātyāyana, and Mānava*, INSA, New Delhi, 1983, p. 169.

[37] Apastamba *śulba sūtra*, 3.2. Sen and Bag, cited above, p. 103. The same thing is repeated in other *śulba sūtra*-s.

early Jain works (such as the *Sūrya prajñāpati*, *sūtra* 20) also use the term *kiñcid viśesādhika* ("a little excess") in describing the value of π and $\sqrt{2}$. Āryabhaṭa (5th c. CE) used the word[38] आसन्न (near value), which term is nicely explained by Nīlakaṇṭha in his commentary,[39] essentially saying that the "real value" (वास्तवीं संख्या) cannot be given.

Taking cognizance of this element of non-representability fundamentally changes arithmetic. This happens, for example, in present-day computer arithmetic, where one is forced to take into account this element of non-representability, for only a finite set of numbers can be represented on a computer. Consequently, even integer arithmetic on a computer can *never* obey the rules of Peano's arithmetic. In the case of real numbers, or floating point computer arithmetic, of course, a mechanical rule is indeed set up for rounding (for instance in the IEEE floating point standard 754 of 1986), and this means that addition in floating point arithmetic is not an associative operation,[40] so that floating point arithmetic would *never* agree with the arithmetic according to any standard formal algebraic structure such as a ring, integral domain, field etc.

In Indian tradition, this difficulty of representation connects to a much deeper philosophy of *śūnyavāda*. On the Buddhist account of the world, the world evolves according to "conditioned coorigination". (A precise quantitative account of what this phrase means to me, and how this relates to current physics is a bit technical, and is available in the literature for those interested in it.[41]) The key point is that there is genuine novelty of the sort that would surprise even God, if he existed. There is no rigid linkage (no Newton's "laws") between present and past, the present is not implicit in the past (and cannot be calculated from knowledge of the past, even by Laplace's demon). Accordingly, *there is genuine change; nothing stays constant.* But how does one *represent* a non-constant, continually changing entity? Note that, on Buddhist thought, this problem applies to *any* entity, for Buddhists believe nothing real can exist unchanged or constant for two instants, so there is no constant entity whatsoever which is permanent or persists unchanged.

This creates a difficulty even with the most common utterances, such as the statement "when I was a boy", for I have changed since I was a boy, and now have a different size, gray hair etc. The linguistic representation however suggests that underlying these changes, there is something constant, the "I" to which these changes happen. Buddhists, however, denied the existence of any constant, unchanging essence or soul for it was neither empirically manifest, nor could it be inferred: the boy and I are really two different individuals with some common

[38] *Gaṇita* 10, trans. K. S. Shukla cited earlier.

[39] *Āryabhaṭīya bhāṣya*, commentary on *Gaṇita* 10, ed. Sambasiva Sastry, cited earlier, p. 56.

[40] For an example of how this happens, see C. K. Raju, "Computers, mathematics education, and the alternative epistemology of the calculus in the Yuktibhasa", *Philosophy East and West*, **51**(3) 2001, pp. 325–62.

[41] The idea is to use functional differential equations of mixed-type to represent physical time evolution. This leads to spontaneity. See C. K. Raju, *Time: Towards a Consistent Theory*, Kluwer Academic, 1994, chp. 5b. Also, C. K. Raju, "Time Travel and the Reality of Spontaneity", *Foundations of Physics* **36** (2006) pp. 1099–1113.

memories. However, while Buddhists accept the reality of impermanence, there is a practical problem of representation in giving a unique name to each individual at each instant. Consider, for example, Ashoka. No one, not even the Buddhists, describe him as Ashoka1, Ashoka2, and so on, with one number for each instant of his life, which cumbersome nomenclature would require some billion different names even on the gross measure of one second as an atomic instant of time. Therefore, for practical purposes, Buddhists recognize the *paucity of names*, and still use a single name to *represent* a whole procession of individuals. This "constancy" of the representation is implicitly understood in the same sense as the constancy of the partial sums of an infinite series: namely, one neglects some small differences as irrelevant to the context. That is, on the Buddhist view of constant change, the customary representation of an individual, used in everyday parlance, as in the statement "when I was a boy", can be obtained only by *neglecting* the changes involved (my size, my gray hair, etc) as inconsequential or irrelevant in the context, and which changes are hence discarded as "non-representable" (for the practical purpose of mundane conversation, in natural language).

So, from the Buddhist perspective of impermanence, mundane linguistic usage necessarily involves such neglect of "inconsequential" things, no matter what one wants to talk about. Note the contrast from the idealistic Platonic and Neoplatonic belief. Plato and Neoplatonists believed in the existence of ideal and unchanging or eternal and constant entities (soul, mathematical truths). Within this idealistic frame, mundane linguistic usage (as in the statement "when I was a boy") admits a simple justification in straightforward sense that change happens to some underlying constant or ideal entity. But this possibility is not available within Buddhism, which regards such underlying ideal entities as fictitious and erroneous, and can, therefore, only speak about non-constant entities, *as if* they were constant. The dot on the piece of paper is all we have, it is the idealization of a geometric point which is erroneous. (Apart from the idealist position, the formalist perspective of set theory also fails, for Buddhist logic is not two valued. But I have dealt with this matter in detail, elsewhere, and we will see this in more detail below.)

Thus, *śunyavāda* or zeroism provides a new way to get over the "fear of small numbers". It was, I believe, Borel who raised the question of the meaning of small numbers such as 10^{-200}. On the *śūnyavāda* perspective, we can discard such numbers as practically convenient. (We have nothing better, no "ideal" or "perfect" way of doing things.) We are not obliged to give a general or universal rule for this, though we can adopt *convenient* practices.

What this amounts to is a realist and fallibilist position. All knowledge (including mathematical knowledge) is fallible.[42] Therefore, when given an exceesively

[42]If mathematical proof is treated as fallible, the criterion of falsifiability would need modification. When a theory fails a test, it is no longer clear what has been refuted: (a) the hypothesis or (b) the deduction connecting hypothesis to consequences. C. K. Raju, "Proofs and refutations in mathematics and physics", in: *History and Philosophy of Science*, ed. P. K. Sen, PHISPC (to appear).

small number we may discard it, as in customary practice, or in computer arithmetic. (Unlike computer arithimetic, where one requires a rule, with human arithmetic, we can allow the "excessive smallness" of the number to be determined by the context.) It is possible, that this leads to a wrong decision. If enough evidence accumulates to the contrary, we revise our decision. It is the search for immutable and eternal truths that has to be abandoned. Such eternal truths are appropriate to religion not any kind of science.

Thus the traditional Indian understanding of mathematics, using zeroism, dispenses with the need for convergence, limits, or supertasks, and rehabilitates the frequentist interpretation of probability, in the sense that it provides a fresh answer to a long-standing philosophical difficulty in the Western tradition.

SUBJECTIVE PROBABILITIES AND THE UNDERLYING LOGIC OF SENTENCES

Probabilities of singular events

Of course, there are other problems with the frequentist interpretation: for example, it does not apply to single events, for which one might want to speak of probability. The classic example is that of a single footprint on a deserted beach (or the origin of life). There is some probability, of course, that someone came in a helicopter and left that single footprint just to mystify philosophers. But, normally, one would regard it as a natural phenomenon and seek a natural explanation for it.

In this context there is an amusing account from Indian Lokāyata tradition, which is the counterpart of the Epicurean perspective in Greek tradition. Here, a man seeking to convert his girlfriend to his philosophical perspective, goes about at night carrying a pair of wolf's paws. He makes footprints with these paws. His aim is to demonstrate the fallibility of inference. He argues that by looking at the footprints, learned people will infer that a wolf was around, and they will be wrong. (We recall that the Lokāyata believed that the only reliable principle of proof was the empirically manifest.)

More seriously, such singular events pose a serious problem today in quantum mechanics, where the "probability interpretation of the wave function" is called into play to explain interference of probabilities exhibited by single objects. A typical illustration of such interference is the two-slit diffraction pattern that is observed even when it is practically assured that electrons are passing through the slit one at a time. Understanding the nature of quantum probabilities has become a major philosophical problem, and we describe below some attempts that have been made to understand this problem by connecting it to philosophies and logics prevalent in ancient Indian tradition.

Quantum mechanics, Boolean algebra, and the logic of propositions

In the 1950's there was a novel attempt to connect the foundations of probability theory to the Jain logic of *syādavāda*, by three influential academicians from India: P. C. Mahalanobis,[43] founder of the Indian Statistical Institute, J. B. S. Haldane,[44] who had moved to that institution, and D. S. Kothari,[45] Chairman of the University Grants Commission. Subsequently, the quasi truth-functional logic used in the structured-time interpretation of quantum mechanics[46] was connected to Buddhist logic.[47]

To understand these attempts, first of all, let us connect them to the more common (Kolmogorov) understanding of probability as a positive measure of total mass 1 defined on a Boolean σ-algebra (usually of Borel sets of a topological space). The common definition typically requires set theory, as we saw above, to facilitate the various supertasks that are required, whether for the construction of formal real numbers as Dedekind cuts, or as equivalence classes of Cauchy sequences, or for the notion of convergence required by the law of large numbers. However, from a philosophical perspective it is more convenient to use *statements* instead of *sets* (though the two are obviously interconnected). Thus, instead of defining probabilities over measurable *sets*, it is more natural to define probabilities over a Boolean algebra of *statements*. This, incidentally, suits the subjectivist interpretation, for the probability of a statement could then be taken to indicate the degree of (general) subjective belief in that statement (or the objective propensity of that statement to be true, whatever that means).

The immediate question, however, is that of the *algebraic structure* formed by these statements. First of all, we can set aside the specifically σ-algebra aspect, for we have already dealt with the notion of convergence and supertasks above. For the purposes of this section we will focus on the Boolean algebra part. Why should probability be defined over a Boolean algebra?

The answer is obviously that if we have a 2-valued logic of sentences, then a Boolean algebra is what we naturally get from the usual notion of "and", "or", "not", which are used to define the respective set-theoretic operations of intersection, union and complementation. What is not obvious is why these "usual" notions should be used, or why logic should be 2-valued.

Quantum mechanics (and especially the problem of the probabilities of singular events in it) provides a specific *empirical* reason to call the Boolean algebra into question. With probabilities defined on a Boolean algebra, joint distributions of

---

[43] P. C. Mahalanobis, 'The Foundations of Statistics (A Study in Jaina Logic)', *Dialectica* **8**, 1954, pp. 95–111; reproduced in *Sankhya*, *Indian Journal of Statistics,* **18**, 1957, pp. 183–94.

[44] J. B. S. Haldane, 'The *Syādavāda* system of Predication', *Sankhya*, *Indian Journal of Statistics*, **18**, 1957, pp. 195–200.

[45] D. S. Kothari, 'Modern Physics and *Syādavāda*', Appendix IV D in in *Formation of the Theoretical Fundamentals of Natural Science* vol. 2 of *History of Science and Technology in Ancient India*, by D. P. Chattopadhyaya, Firma KLM, Calcutta, 1991, pp. 441–48.

[46] C. K. Raju, *Time: Towards a Consistent Theory*, cited above, chp. 6B, "Quantum Mechanical Time".

[47] C. K. Raju, *The Eleven Pictures of Time*, Sage, 2003

random variable are assured to exist. This is, however, known to *not* happen in quantum mechanics. (We will not go into details, since our primary concern here is with Indian tradition, and not quantum mechanics. However, this author has explained the detailed relation to quantum mechanics elsewhere, at both a technical[48] and a non-technical level.[49]) The Hilbert space formulation of quantum mechanics starts with the premise that quantum probabilities cannot be defined on a Boolean algebra, since joint distributions do not exist. The appropriate algebraic structure is taken to be that of the lattice of subspaces (or projections) of a Hilbert space (although there are numerous other opinions about what the exact algebraic structure ought to be).

The usual definition of a random variable as a measurable function actually requires only the *inverse* function, which is a homomorphism which preserves the algebraic structure. In the Hilbert space context, this definition of a random variable as a homomorphism (on a lattice, not an algebra) naturally leads one to identify random variables with projection-valued measures (spectral measures). By the spectral theorem, such measures correspond to densely-defined, self-adjoint operators in this Hilbert space. Since the lattice of projections is non-distributive, these random variables do not admit joint distributions. This corresponds to the more common assertion ("uncertainty principle") that dynamical (random) variables (self-adjoint operators) which do not commute cannot be simultaneously measured.

To return to the question of logic, unlike in India, where different types of logic have been in existence for over 2500 years, from pre-Buddhist times,[50] the West took cognizance of the existence of logics that are not 2-valued, only from the 1930's onwards, starting with Lukaciewicz who proposed a 3-valued logic, where the truth values could be interpreted as "true", "false", and "indeterminate". Could such a 3-valued logic account for quantum probabilities? This question was first investigated by Reichenbach, in an unsuccessful interpretation of quantum mechanics.

The 3 Indian academics mentioned above also interpreted the Jain logic of *syādavāda* (perhaps-ism) as a 3-valued logic (Haldane), and explored 3-valued logic as a philosophical basis for formulating probabilities (Mahalanobis), and interpreting quantum mechanics (Kothari). Haldane's idea related to perception. With repeated experiments, something on the threshold of perception (such as a sound) may be perceptible sometimes, and sometimes not. In such cases, the "indeterminate" truth value should be assigned to the statement that the "something" is perceptible. Mahalanobis' idea was that this third truth value was already a rudimentary kind of probability, for it expressed the notion of "perhaps". Kothari's idea was to try and explain quantum mechanics on that basis (though he overlooks Reichenbach's earlier unsuccessful attempt).

---

[48]C. K. Raju, *Time: Towards a Consistent Theory*, cited above, chp. 6B, "Quantum Mechanical Time".

[49]C. K. Raju, *The Eleven Pictures of Time*, Sage, 2003.

[50]B. M. Baruah, *A History of Pre-Buddhistic Indian Philosophy*, cited above.

Buddhist and quasi truth-functional logic

While Haldane's interpretation is clear enough within itself, it is not clear that it accurately captures the logic of *syādavaāda*. Thus, the Jain tradition grew in the vicinity of the Buddhist tradition (the Buddha and Mahavira were contemporaries). However, Buddhist logic is not 3-valued. For example, in the Dīgha Nikāya, the Buddha asserts the existence of 4 alternatives (*catuṣkoṭi*): (1) The world is finite; (2) the world is not finite (= infinite); (3) the world is both finite and infinite; and (4) the world is neither finite nor infinite.[51]

This logic of 4-alternatives does *not* readily fit into a multi-valued truth-functional framework. Especially, the third alternative, which is of the form $A \wedge \neg A$, is a contradiction within 2-valued logic, and difficult to understand even within the frame of 3-valued logic, where it cannot ever be "true"'. The reason why 3-valued logic is not appropriate for quantum probabilities is roughly this: in the case of the two-slit experiment, what is being asserted is that it is true that the electron passed through *both* slit A *and* slit B, and not that in reality it passed through only one slit, but we do not *know* which slit it passed through. What is being asserted is that we know that Schrödinger's cat is both alive and dead, as in the third alternative above, and not that it is either alive or dead, but we do not know which is the case.

However, the 3$^{\text{rd}}$ alternative of the Buddhist logic of 4 alternatives (*catuṣkoṭi*) makes perfect sense with a quasi truth-functional logic. The standard semantics here uses the Tarski-Wittgenstein notion of logical "world", as "all that is the case". On this "possible-world semantics" one assigns truth values (either true or false) to all atomic statements: such an assignment of truth values represents the possible facts of the world (at one instant of time), or a "possible world". This enables the interpretation of modal notions such as possibility and necessity: a statement is "possible" if it is true in *some* possible worlds, and "necessary" if it is true in *all* possible worlds (tautology). In fact, Haldane appeals to precisely this sort of semantics, in his interpretation of Jain logic, except that his "worlds" are chronologically sequential. Thus, A is true at one instant of time, while not-A is true at *another* instant of time—there is nothing paradoxical about a cat which is alive now, and dead a while later. However, this, as we have observed, is not appropriate to model the situation depicted by quantum mechanics.

With quantum mechanics what we require are multiple logical worlds attached to a *single* instant of time. Parallel computing provides a simple and concrete desktop model of this situation, with each processor represented by a separate logical world. The meaningfulness of a quasi truth-functional logic is readily grasped in this situation where multiple (logical) worlds are chronologically simultaneous and not sequential, for this allows a statement to be simultaneously both true and false. That is, with multiple (2-valued) logical worlds attached to a *single* instant

[51] Brahmajāla sutta of the Dīgha Nikāya. (Hindi trans.) Rahul Sānkrityāyana and Jagdish Kāshyapa. Delhi: Parammitra Prakashan, 2000, pp. 8–9; (English trans.) Maurice Walshe, Boston: Wisdom Publication, 1995, pp. 80–81. For a more detailed exposition, see C. K. Raju, "Logic", in *Encyclopedia of Non-Western Science, Technology and Medicine*, Springer, 2008.

of time, it is meaningful to say that A is true in one world while $\neg A$ is *simultaneously* true in another. So, a statement may be simultaneously both true and false, without trivializing the theory, or making it inconsistent. From our immediate perspective, the important thing is this: such a quasi truth-functional logic leads on the one hand to an algebraic structure appropriate to quantum probabilities, which structure is not a Boolean algebra.[52] On the other hand, Buddhist logic (*catuṣkoṭi*) naturally admits an interpretation as a quasi truth-functional logic. Thus, Buddhist logic (understood as quasi truth-functional) leads to just the sort of probabilities that seem to be required by quantum mechanics.

This quasi truth-functional logic, corresponding to simultaneous multiple worlds, is not a mere artificial and *post-facto* construct imposed on either quantum mechanics or Buddhist thought. From the viewpoint of physics, quasi truth-functional logic arises naturally by considering the nature of time. This is best understand through history. Hoping to make "rigorous" the imported calculus, and the notion of derivative with respect to time required for his "laws", Newton made time metaphysical ("*absolute*, *true*, and *mathematical* time" which "flows equably *without relation to anything external*"[53]). Eventually, this intrusion of metaphysics and religious belief into physics had to be eliminated, from physics, through a revised *physical* definition of the measure of time; that directly led to the special theory of relativity.[54] A correct mathematical understanding of relativity, shows that physical time evolution must be described by functional differential equations (and not ordinary differential equations or partial differential equations). The further elimination of the theological understanding of causality in physics makes these functional differential equations of mixed-type. The resulting picture of physical time evolution[55] is remarkably similar to the core Buddhist notion of "conditioned coorigination": where the future is *conditioned* by the past, but *not* decided by it. There is genuine novelty. Thus, the relation of the quasi truth-functional logic to the revised notion of time, in physics, parallels the relation of Buddhist logic to the Buddhist notion of "conditioned coorigination" (*paticca samuppāda*). Note that this last notion differs from the common notion of "causality" used in Western thought, with which it is commonly confounded.

Of course, formal Western mathematics (and indeed much of Western philosophy) is likely to be a long-term casualty of any departure from 2-valued logic. In fact, the very idea that logic (or the basis of probability) is not culturally universal, and may not be empirically certain, unsettles a large segment of Western thought,

[52]See the main theorem in C. K. Raju "Quantum mechanical Time", cited above.

[53]I. Newton, *The Mathematical Principles of Natural Philosophy*, A. Motte's translation revised by Florian Cajori, Encyclopedia Britannica, Chicago, 1996, p. 8.

[54]For an exposition of Poincaré's philosophical analysis of the notion of time which led to the special theory of relativity, see C. K. Raju, "Einstein's time", *Physics Education* (India), **8**(4) (1992) pp. 293–305. A proper clock was defined by *postulating* the velocity of light to be a constant. This had nothing to do with any experiment. See, also, C. K. Raju, "The Michelson-Morley experiment", *Physics Education* (India) **8**(3) (1991) pp. 193–200.

[55]C. K. Raju, "Time travel and the reality of spontaneity", *Found. Phys.* **36** (2006) pp. 1099–1113.

and its traditional beliefs about induction and deduction.

INDEX