Probablity

C. K. Raju

The mathematical theory of probability begins with the theory of permutations and combinations, needed to calculate probabilities in games of chance, such as dice or cards. The earliest account of this theory is found in India. This theory is tied to the theory of the Vedic metre (and the theory of Indian music, in general).

The Vedic and post-Vedic composers used combinations of two syllables called *guru* (deep, long) and *laghu* (short). The earliest written account of this theory of metre is in Piñgala's *Chandahsūtra* (-3^{rd} c. CE), a book of aphorisms (*sūtra*-s) on the theory of metre (*chanda*). To calculate all possible combinations of these two syllables in a metre containing n syllables, Piñgala gives the following rule[1] (which explicitly makes use of the symbol for zero). "(Place) two when halved;" "when unity is subtracted then (place) zero;" "multiply by two when zero;" "square when halved." In a worked example, Dutta and Singh[2] show how this rule leads to the correct figure of 2^6 possibilities for the Gāyatrī metre with 6 syllables.

Piñgala's commentator the 10^{th} c. CE Halayudha clarifies that this involves the binomial expansion. Thus, in a 3-syllabic metre with two underlying syllables, *guru* and *laghu*, 3 *guru* sounds will occur once, 2 *gurus* and 1 *laghu* will occur twice, as will 1 *guru* and 2 *laghus*, while 3 *laghus* will occur once. Symbolically $(g + l)^3 = g^3 + 3g^2l + 3gl^2 + l^3$. To generalize this to the case of n underlying syllables, Halayudha explains the *meru-prastāra* (pyramidal expansion) scheme for calculation,[3] which is identical to "Pascal's" triangle which first appeared in Europe over a thousand years after Piñgala and about a century before Pascal (on the title page of the *Arithmetic* of Apianus) and in China in the 14^{th} c.[4] An example, using the *Gāyatrī* meter is also found in Bhaskara's *Līlāvatī*.[5] Stock Western histories of mathematics (such as that by Smith[6]) are unreliable and wrongly state that no attention was paid in India to the theory of permutations and combinations before Bhaskara II (12^{th} c. CE).

Although this theory of permutations and combinations is built into the Vedic metre, the earliest known written account actually comes from even before Piñgala, and is found in the -4^{th} c. Jain *Bhagwatī Sūtra*. Permutations were called *vikalpa-gaṇita* (the calculus of alternatives), and combinations *bhanga*. The text works out the number of combinations of n categories taken 2, 3 etc. at a time.

From the earliest Vedic tradition, there is a continuous series of manuscripts linking the first accounts of permutations and combinations with those of Bhaskara II (12^{th} c.), and later commentaries on his work, up to the 16^{th} c. CE, such as the *Kriyākramkarī*. (It is these latter texts which arrived in Europe and first brought the mathematical theory of probability to Europe, as detailed later on, through Pascal, Fermat etc.) Thus,[7] the surgeon Suśruta (-2^{nd} c. CE) in his compendium (*Suśruta-samhitā*) lists the total number of flavours derived from 6 flavours taken 1 at a time, 2 at a time, and so on. Likewise, Varāhamihira (6^{th} c.) states the number of perfumes that can be made from 16 substances mixed in 1, 2, 3, and 4 proportions. Similar examples are found in the *Pāṭīgaṇita* (Slate Arithmetic) of Śridhar (10^{th} c.), a widely used elementary school-text, as its name suggests, Mahavira's (8^{th} c.) *Gaṇita Sāra Saṁgraha,* and Bhaskara II (*Līlāvatī*) etc. Bhaskara mentions that this formula has applications to the theory of metre, to architecture, medicine, and *khaṇḍameru* ("Pascal's triangle"). In these later texts, one finds explicitly stated formulae for permutations and combinations.

For example, to calculate $\binom{n}{r}$ values, Śridhar, in his text on slate-arithmetic[8] (*Pāṭigaṇita*), gives the following rule.

एकाद्युत्तरविधिना रस विन्यासे विलोमतो गुणयेत् ।
पूर्वेंण परं क्रमशो रूपादिचयैर्हरेर्विभ्जेत् ॥

This translates as follows (*Pāṭigaṇita*, 72, Eng. p. 58)

> Writing down the numbers beginning with 1 and increasing by 1 up to the (given) number of savours in the inverse order, divide them by the numbers beginning with 1 and increasing by 1 in the regular order, and then multiply successively by the preceding (quotient) the succeeding one. (This will give the number of combinations of the savours taken $1, 2, 3, ...,$ all at a time respectively.)[9]

Thus, in the case of 6 savours, one writes down the numbers 1 to 6 in reverse order

$$6, 5, 4, 3, 2, 1$$

These are divided by the numbers in the usual order, to get the quotients

$$\frac{6}{1}, \frac{5}{2}, \frac{4}{3}, \frac{3}{4}, \frac{2}{5}, \frac{1}{6}.$$

Then, according to the rule, the number of combinations of savours taken 1 at a time, 2 at a time, etc., up to all at a time are respectively

$$\frac{6}{1}, \frac{6}{1}\times\frac{5}{2}, \frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}, \text{ etc.}$$

Bhaskara II illustrates these formulae by asking for the total number of 5 digit numbers whose digits sum to 13.[10]

पंचस्थानस्थितैरंकैर्यद्यद्योगस्त्रयोदश ।
कतिभेदा भवेत्संख्या यदि वेत्सि निगद्यताम् ॥

Almost all Western histories of probability incorrectly state that the mathematical theory of probability began with a European dispute over gambling in the 17th c. CE. This account intrinsically lacks credibility because, given that gambling is so ancient, it is hard to understand why a mathematical theory related to it should have developed so late. One readily understands why this happened late in Europe, since Greek and Roman numerals are not suited to anything beyond elementary arithmetic, and cannot even represent fractions, and arithmetical skills became widespread in Europe only after elementary Indian arithmetic was first introduced in the Jesuit syllabus ca. 1572.

But mathematics related to games of chance did develop elsewhere. In fact, elementary Indian mathematical texts give us accounts of gambling strategies and methods of calculating gain or loss. The relevance of these weighted averages to gambling was understood. For example, the *Gaṇita Sāra Saṃgraha*[11](268.5, 273.5) gives the example of "Dutch bets" mentioned by Hacking.[12] This is "a rule to ensure profit (in gambling) regardless of victory or loss", a method of riskless arbitrage, in short. The text illustrates the rule with an example, where "a great man knowing mantra and medicine sees a cockfight in progress. He talks to the owners of the birds separately in a mysterious way. He tells one that 'if your bird wins, you give me the amount you bet, and if it loses, I will give you $\frac{2}{3}$ of that amount'. Then he goes to the owner of the other bird where on those same conditions he promised to pay $\frac{3}{4}$ of the amount. In either case, he earned a profit of only 12 pieces of gold. O mathematician, blessed with speech, tell me how much money did the owner of each bird bet."

It is also interesting to note that this involves the formula for "mixtures" or weighted averages, exactly the formula used to calculate mathematical expectation (or, say, the expected value of a lottery ticket) in probability theory today. There are other more complicated betting strategies showing that standard Western accounts that the mathematics of gambling began in Europe in the 17th c. are completely misplaced; that is when it began in Europe, whereas it began in the world a few thousand years earlier.

Needless to say, the game of dice itself is very ancient in India. The earliest known textual account of gambling is found in the *akṣa sūkta* or hymn on dice in the Ṛgveda (10.2.34). The long hymn begins by comparing the pleasure of gambling with the pleasure of drinking soma!

"There is enjoyment like the soma in those dice".[13] (सोमस्येव मौजवतस्यमद्वो)

It goes on to describe how everyone avoids a gambler, like an old man avoids horses, even his mother and father feign not to recognize him, and he is separated from his loving wife. Many times the gambler resolves to stay away, but each time the fatal attraction of the dice pulls him back. With great enthusiasm he reaches the gambling place, hoping to win, but sometimes he wins and sometimes his opponent does. The dice do not obey the wishes of the gambler, they revolt. They pierce the heart of the gambler, as easily as an arrow or a knife cuts through the skin, and they goad him on like the ankuṣ, and pierce him like hot irons. When he wins he is as happy as if a son is born, and when he loses, he is as if dead. The 53 dice dance like the sun playing with its rays, they cannot be controlled by the bravest of the brave, and even the king bows before them. They have no hands, but they rise and fall, and men with hands lose to them. The gambler's wife remains frustrated and his son becomes a vagabond. He always spends his night in other places. Anyone who lends him money doubts that he will get it back. The gambler who arrives in the morning on a steed leaves at night without clothes on his back. [Such is the power of dice!] O Dice, I join the ten fingers of my hands and bow to the leader among you!

The mathematics of probability was not restricted to games of chance in India: at a very early stage it was linked to statistics or sampling theory, used, for example, to count the number of fruits in a tree. The first account of it is found in the Mahabharata epic, traditionally dated to the beginning of the Kali era (-3102 or about 5000 years ago). The Mahabharata war, itself, is precipitated when the heroes (Pandava-s) lose their kingdom and dignity during a game of dice (द्युत क्रीडा). Yudhishthira, the Pandava leader, who loses the game of dice, realizes that the game is not fair, and says so: "निक्रितिर्देवनं पापं न क्षात्रोत्र पराक्रमः" (Mbh, Sabhā Parva, 59.5). That is, "Deceitful gambling is sinful, there is no Kshatriya valor in it." Nevertheless, he continues to play, since he cannot prove the dice are loaded, and, as a Kshatriya, he would be dishonored if he refuses to participate in an enterprise involving risk.

When Yudhishthira is banned to the forest, and ruing his condition, a passing sage consoles him by telling him the romantic story of Nala and Damayanti. Nala, too, looses his kingdom in a deceitful game of dice, and looses even his last clothes in a forest where he abandons his wife Damayanti, since he is unable to support her. Someone advices him to go to Ṛtuparṇa, king of Ayodhya, knowledgeable in the science of dice. Accordingly, Nala disguises himself and takes up a job as a charioteer with Ṛtuparṇa. Damayanti somehow finds her eventful way home, and sends out search parties for Nala. Eventually, she locates him, and to test whether he really is Nala, she announces her second marriage through a *swayamvara* (ceremony to choose a husband). Ṛtuparṇa wants to wed Damayanti and rushes to Vidarbha. The distance from Ayodhya to Vidarbha being so large, only an accomplished charioteer like Nala can reach in time, so Ṛtuparṇa naturally turns to him.

On the way they stop for a moment under a Vibhītaka tree[14]—it is significant that the nuts of this very tree, with five faces, were used in the ancient Indian game of dice. Ṛtuparṇa cannot resist showing off his knowledge of mathematics and says: the number of fruits in the two branches of the tree is 2095, count them if you like. Nala wants to know how this was done, and threatens to stop and count. He also offers his knowledge of horses in return. Under tight time-pressure, Ṛtuparṇa succumbs and explains that it is done by counting a portion of one branch. (The story has a happy ending and Nala gets back both his wife and his kingdom.) However, what it tells us is that this knowledge of the mathematics of dice, though very ancient, was regarded as valuable, and kept a secret to be passed on from one person to another, and not put in common texts.

Lastly, there is the question of continuous probabilities, which are today defined using Kolmogorov's axioms or measure theory. This approach is not really satisfactory since our empirical understanding of probability is based on relative frequency and the so-called "law of large numbers". If a coin is tossed 100 times and comes up heads 48 times the relative frequency of heads is $\frac{48}{100} = 0.48$. On the next 100 tosses, the relative frequency may change to 0.47. However, relative frequency is not the probability, which could be 0.5 if the coin is unbiased. The belief is that, as the number of coin tosses becomes large, the relative frequency will be closer to the "true" probability. But "closer" in what sense? Suppose, after a 1000 tosses, the relative frequency is 0.49. Can we be sure that after 2000 tosses it would not be 0.47? We cannot. In fact, it can even happen that the next 1000 tosses all result in tails. All we can say (and all that the "law of large numbers" enables us to say) is that it is *unlikely* that this will happen. That is, the probability that this will happen is small. This state of affairs means that empirical relative frequency cannot be used to define axiomatic probability. The issue of the failure of this frequentist interpretation of probability, and the way it is resolved by zeroism is taken up in a more advanced article.[15]

Notes

[1] *Chandahsūtra* viii.28. Ed. Sri Sitanath, Calcutta, 1840. Cited by Dutta and Singh, vol. 1, p. 76.

[2] B. B. Dutta and A. N. Singh, *History of Hindu Mathematics*, Asia Publishing House, 1962, vol. 1, pp. 75–76.

[3] A. K. Bag, *Mathematics in Ancient and Medieval India*, Chaukhambha Orientalia, Varanasi, 1979, pp. 189–93.

[4] Joseph Needham, *The Shorter Science and Civilisation in China*, vol. 2 (abridgement by C. A. Ronan). Cambridge University Press, 1981, p. 55.

[5] Bhaskara, *Līlāvatī*, trans. K. S. Patwardhan, S. A. Naimpally, and S. L. Singh, p. 102. The verse is numbered differently in different manuscripts. K. V. Sarma in his critical edition of the 16th c. southern commentary *Kriyākramakarī* (VVRI, Hoshiarpur, 1975) on the *Līlāvatī*, gives this as verse number 133, while the other cited source has given it as verse number 121.

[6] D. E. Smith, *History of Mathematics*, Dover Publications, 1958, vol. 2, p. 502.

[7] A. Bag, *Mathematics in Ancient and Medieval India*, cited above, p. 188.

[8] Śridhar, *Pāṭigaṇita*, 72, ed. & trans. K. S. Shukla, Dept. of Mathematics and Astronomy, Lucknow University, 1959, Sanskrit, p. 97.

[9] *Pāṭigaṇita of Śridhar*, trans. K. S. Shukla. As he points out, similar articulations are found in the *Gaṇita Sāra Saṃgrah* of Mahavira, vi.218, *MahāSiddhānta* of Āryabhaṭa 2, xv, 45–46 etc.

[10] *Līlāvatī of Bhāskarācārya*, trans. Patwardhan et al., p. 181. They give the number of this verse as 276, whereas, in K. V. Sarma's critical edition of *Kriyākramkarī*, a commentary on the *Līlāvatī*, this is at 269, p. 464.

[11] *Gaṇitasāra-Saṃgraha*, (Hindi trans. L. C. Jain), Jain Samskrti Samraksha Sangh, Sholapur, 1963, pp. 159–60.

[12] Ian Hacking, *The Emergence of Probability: A Philosophical Study of Early Ideas about Probability, Induction and Statistical Inference*, Cambridge University Press, 1975, pp. 6–9. Hacking uses the English translation of *Gaṇita Sāra Saṃgraha*, trans. M. Rangacharya (1912), pp. 162–3.

[13] Ṛgveda, 10.2.34.1

[14] Mahabharata, *van parva*, 72, trans. K. M. Ganguly, 1883–1896, Book 3, pp. 150–51.

[15] C. K. Raju, "Probability in Ancient India", *Handbook of the Philosophy of Science*, vol 7. *Philosophy of Statistics*, ed. Prasanta S. Bandyopadhyay and Malcolm R. Forster. General Editors: Dov M. Gabbay, Paul Thagard and John Woods. Elsevier, 2011, pp. 1175–1196.