Raju, C. K. ‘Calculus’. In Encyclopedia of Non-Western Science, Technology and Medicine, edited by Helaine Selin, 1010–15. Dordrecht: Springer, 2016.

Calculus

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The Indian origin of infinite series, found in widely distributed texts, has long been publicly known to Western scholars (Whish 1832). Recent research (Raju 2007) has shown that these Indian developments really did amount to the calculus. (This brings to the forefront various epistemological issues, and the very philosophy of mathematics taken for granted in Western discourse.) This research has also pushed the historical origin of the calculus in India much further back, to the 5th c. CE Āryabhaṭa, and his method of obtaining sine values by numerically solving the corresponding differential equation using a finite difference technique.

The infinite series had originated in India by the 14th c. An explicit formula for the sum of an infinite (*anantya*) geometric series is given by the 15th-16th c. Nīlakaṇṭha in his *Āryabhaṭīyabhāṣya*.(Sastri 1970, commentary on *Gaṇita* 17, p. 142.)

एवं यस्तुल्यच्छेदपरभागपरम्पराया अनन्ताया अपि संयोगः
तस्यानन्तानामपि कल्प्यमान्स्य योगस्याद्यावयविनः
परम्परांशच्छेदादेकोनच्छेदांशसाम्यं सर्वत्रापि समानमेव।

which may be translated:(cf. Sarma 1972, p. 17)

> The sum of an infinite [*anantya*] series, whose later terms (after the first) are got by dividing the preceding one by the same divisor everywhere, is equal to the first term multiplied by the common divisor, and divided by one less than the common divisor.

That is $a + \frac{a}{d} + \frac{a}{d^2} + \cdots = \frac{ad}{d-1}$. (It is assumed that the divisor $d > 1$, so that the common ratio is less than 1.) This is obviously the earliest textual source in the world for a formula for a properly infinite sum.

Was this infinite sum based on a “rigorous” notion of limits? In fact, it was based on something superior. Indian mathematicians used a different number system involving, what would today be called "non-Archimedean" arithmetic, *and* a different epistemology, which allowed discarding of infinitesimals (see also ZEROISM).

In terms of formal mathematics, the Indian approach may be understood as follows. Ever since algebra began (with Brahmagupta) Indian mathematicians referred to polynomials as "unexpressed (*avyakta*) numbers". This led naturally to "unexpressed fractions", or what formal mathematicians would today call the non-Archimedean ordered field of rational functions.(For an elementary construction of this field, see Moise 1968, appendix.)

A non-Archimedean field contains a number x such that $x > n$ for every integer n. Such a number may be called infinite. Being a field, it also contains the multiplicative inverse of the number which satisfies $0 < \frac{1}{x} < \frac{1}{n}$ for every integer n, so that $\frac{1}{x}$ may be called infinitesimal. The existence of infinitesimals means that limits are not unique, and one must discard infinitesimals. ("Infinitesimal" does *not* mean the same thing as "imperceptible" or "intangible", e.g. $\frac{3}{n}$ or $\frac{n}{n^2}$ are infinitesimals.) The traditional Indian process of summing infinite series by discarding of infinitesimals, as used in deriving the above formula, is equivalent to limits by order-counting (Raju 2007, chp. 3).

Discarding infinitesimals is related to discarding small numbers, but clearly goes a step beyond it. Small numbers were traditionally discarded, from at least 2500 years ago, since the days of the *śulba sūtra* to provide "approximate" values of $\sqrt{2}$, π etc. The terms then used were *saviśeṣa* (with something remaining) (Baudhāyana *śulba sūtra*, 2.12l, Sen and Bag 1983, p. 169), *anitya*(impermanent) (Apastamba *śulba sūtra*, 3.2, Sen and Bag 1983) in the *śulba sūtra* and *āsanna* (near value) by Āryabhaṭa (Gaṇita, 10, Shukla and Sarma 1976). This was similar (but not identical) to the way one today has no choice but to discard small numbers in computer (floating point) arithmetic, since a computer cannot physically implement formal "real" numbers obtained as metaphysical limits.

Further since Āryabhaṭa (5th c.), at least, an algorithm was available for square root extraction, so there was a recursive process of generating an infinite series for, say, $\sqrt{2}$. A recursive series for difficult fractions, similar to continued fractions, was also used since Brahmagupta.(For full details, and for the use of this fraction-series expansion in developing later-day infinite series, see Raju 2007, chp. 3.) Summing an infinite series, as Nīlakaṇṭha did, by discarding infinitesimals, or order counting, clearly goes beyond the much older process of discarding small quantities by truncation of infinite series or halting those recursive procedures after a finite number of steps.

Further, one should not just assume that metaphysical limits or Western metaphysical ways of handling infinity are automatically superior. Thus, the use of non-Archimedean arithmetic (plus zeroism) leads to a notion of derivative superior to the present-day college-text definitions of the derivative as a limit, for the college-text definition fails with a discrete number system like floating point numbers on a computer, or with discontinuous functions. While the Schwartz theory does allow a discontinuous function to be differentiated, Schwartz distributions cannot be readily multiplied, therefore that notion of derivative cannot be used for non-linear differential equations of physics (Raju 1989).

The whole problem of how to define the derivative or the notion of probability has a natural and very simple resolution using Indian "non-Archimedean" arithmetic and the related philosophy of zeroism, a solution which is better and far simpler than the resolution using non-standard analysis, as was used earlier (Raju 2007, appendix).

Further, it is clearly wrong to assume that the dominant philosophy of formalism, even though globalised by colonialism, represents a universal philosophy of mathematics, or is the only correct way to do mathematics. Even the logic underlying mathematical proofs is not universal (Raju 2001; see also, article on LOGIC). Formalist philosophy hence involves a biased metaphysics (Raju 2011b).

Calculus can also be taught with a different philosophy which follows its actual historical development and makes it easy to learn. (Thus, over the last few years, "decolonised" courses on "calculus without limits" have been taught with significant success to 8 groups in four universities in three countries; Raju 2011c, Raju 2012.)

Indian mathematicians long knew about *finite* geometric series, which they called *guṇottara saṅkalitā* or *guṇottara śreḍhī* (multiplicative series), and methods of summing the finite geometric series were a part of the elementary school curriculum from at least 1300 years ago, and are found in numerous elementary texts such as the *PāṭiGaṇita* of Śrīdhara (Śrīdhara, Shukla 1959, trans. p. 75.) Indeed, summing a finite multiplicative series was a very common commercial problem found in a variety of other sources such as the *GaṇitaSāra Saṁgraha* of Mahāvīra (Jain 1963) *MahāSiddhānta* of Āryabhaṭa II (Dwivedi 1900) and the *Līlāvatī* of Bhaskara II etc. (rule 126 et. seq; the numbering differs between Colebrooke 1816 and the critical edition of Sarma 1975).

In fact, the place value system lends itself naturally to "geometric" (multiplicative) series, and the first such series is found thousands of years earlier in the Yajurveda (17.2), and in the Lalita Vistara Sutra (chp. 12) where it goes up to large numbers like 10^{53} (named *tallakṣaṇa*).

Of what use were these infinite series? An infinite series expansion for the sine and cosine (today called the "Taylor" series, after a pupil of Newton) was used by Madhava of Sangamagrama (1340 CE) to calculate precise sine and cosine values, by truncating the series to 11th and 12th order polynomials. The infinite series expansion is found in two verses stated in various texts, such as the *TantrasaṅgrahaVyākhyā/Yuktidīpikā* (2.440–443) (Sarma 1977).

निहत्य चापवर्गेण चापं तत्तत्फलानि च ।
हरेत् समूल्युग्वर्गैस्त्रिज्यावर्गहतैः क्रमात् ॥ ४४० ॥
चापं फलानि चाधोऽधो न्यस्योपर्युपरित्यजेत् ।
जीवाप्त्यै, संग्रहोऽस्यैव विद्वान् इत्यदिना क्रितः ॥ ४४१॥

This verse may be translated:

> Multiply the arc by the square of the arc, and take the result of repeating that [any number of times]. Divide [each of the above numerators] by the squares of successive even numbers increased by that number [lit. the root] and multiplied by the square of the radius. Place the arc and the successive results so obtained one below the other, and subtract each from the one above. These together give the *jīvā*, as collected together in the verse beginning with *"vidvān"* etc.

In present-day mathematical terminology, the above passage says the following. Let r denote the radius of the circle, let s denote the arc and let t_n denote the nth expression obtained by applying the rule cited above. The rule requires us to calculate as follows.

1. Numerator: multiply the arc s by its square s^2, this multiplication being repeated n times to obtain $s \cdot \prod_{1}^{n} s^2$.

2. Denominator: multiply the square of the radius, r^2, by $[(2\,k)^2 \;+\; 2\,k]$ ("the squares of successive even numbers increased by that number") for successive values of k, repeating this product n times to obtain $\prod_{k=1}^{n} r^2\,[(2k)^2 \;+\; 2k]$.

Thus, the nth iterate is obtained by

$$t_n \;=\; \frac{s^{2n} \;\cdot s}{(2^2 \;+\; 2) \;\cdot\; (4^2 \;+\; 4) \cdot \;\cdots\; \cdot [(2n)^2 \;+\; 2n] \;\cdot\; r^{2n}}. \tag{1}$$

The rule further says:

$$\text{jīvā} \;=\; s - t_1 \;+\; t_2 - t_3 \;+\; t_4 - t_5 \;+\; \ldots \tag{2}$$

$$= \; s \;-\; \frac{s^3}{r^2 \cdot (2^2+2)} \;+\; \frac{s^5}{r^4(2^2+2)(4^2+4)} \;-\; \cdots . \tag{3}$$

Substituting
(1) jīvā $\equiv\; r \sin\theta$,
(2) $s \;=\; r\,\theta$, so that $s^{2n\,+\,1}\,/\,r^{2n} \;=\; r\,\theta^{2n+1}$, and noticing that
(3) $[(2k)^2 + 2k] \;=\; 2k \cdot (2k+1)$, so that
(4) $(2^2+2)(4^2+4)\cdots[(2n)^2 \;+\; 2n] \;=\; (2n\;+1)!$,
and cancelling r from both sides, we see that this is entirely equivalent to the well-known infinite series

$$\sin\theta \;=\; \theta \;-\; \frac{\theta^3}{3!} \;+\; \frac{\theta^5}{5!} \;-\; \frac{\theta^7}{7!} \;+\; \cdots . \tag{4}$$

A similar rule gives another infinite series for *śara*, corresponding to the infinite series for cosine. A similar series was available for the inverse arctangent function, and various other infinite series were developed in the process of accelerating convergence of these series. (Raju 2007, chp. 3.)

The above infinite series was used to device a numerically efficient formula (Raju 2007, chp. 3) which finally leads to the table of 24 sine values accurate to the third sexagesimal minute (*tatparā*), and stated in reverse sexagesimal *kaṭapayādi* notation. (For a detailed explanation of this *kaṭapayādi* system, and the precise accuracy of the values, see Raju 2007, chp. 3.) The first few and last few lines of the verse are

श्रेष्ठं नाम वरिष्ठानां हिमाद्रिर्वेदभावनः ।
तपनो भानुसूक्तज्ञो मध्यमं विद्धि दोहनम् ॥
धिगाज्यो नाशनं कष्टं छन्नभोगाशयाम्बिका ।
म्रिगाहारो नरेशोऽयं वीरो रणजयोत्सुकः ॥
...
छायालयो गजो नीलो निर्मलो नास्ति सत्कुले ।
रात्रौ दर्पणमभ्राङ्गं नागस्तुङ्गनखो बली ॥

धीरो युवा कथालोलः पूज्यो नारीजनैर्भगः।
कन्यागारे नागवल्ली देवो विश्र्वस्थली भृगुः॥
तत्परादिकलान्तास्तु महाज्या माधवोदिताः।
स्वस्वपूर्वविशुद्धे तु शिष्टास्तत्खण्डमौर्विकाः॥ २.९.५॥

This corresponds to the values

Table 1: Mādhava's sine values

No.	Kaṭapayādi	kalā (′)	vikalā(″)	tatparā(‴)
1	श्रेष्ठं नाम वरिष्ठानां	224	50	22
2	हिमाद्रिर्वेदभावनः	448	42	58
3	तपनो भानुसूक्तज्ञो	670	40	16
4	मध्यमं विद्धि दोहनम्	889	45	15
...	...	...	...	...
21	धीरो युवा कथालोलः	3371	41	29
22	पूज्यो नारीजनैर्भगः	3408	20	11
23	कन्यागारे नागवल्ली	3430	23	11
24	देवो विश्र्वस्थली भृगुः	3437	44	48

The accuracy of these values in decimals is given in Table 2.

Table 2: Accuracy of Mādhava's sine table.

No.	Mādhava's sine value	Difference
1	0.0654031452	0.0000000160
2	0.1305262297	0.0000000375
3	0.1950903240	0.0000000020
4	0.2588190035	−0.0000000416
...	...	...
...	...	...
21	0.9807852980	0.0000000176
22	0.9914448967	0.0000000353
23	0.9978589819	0.0000000587
24	1.0000000000	0.0000000000

The actual calculation requires the value of the number today called π. Madhava had obtained the value accurate to 12 decimal places. This was obtained by accelerating the convergence of various slowly convergent series such as the infinite series today wrongly called “Leibniz series”. Nīlakaṇṭha, in his *ĀryabhaṭīyaBhaṣya*, in his commentary on *Gaṇita* 10 (Sastri 1930, p. 56), described the subtle value of π given by Madhava of Sangamgrama as follows:

सङ्गमग्रामजो माधवः पुनरत्यासन्नां परिधिसंख्यामुक्तवान् –

विबुधनेत्रगजाहिहुताशनत्रिगुणवेदभवारणबाहवः।
नवनिखर्वमिते वृतिविस्तरे परिधिमानमिदं जगदुर्बुधाः॥

The numbers in this verse are according to the old *bhūta saṁkhyā* system, which uses word numerals. Thus, *netra* means 2 because one has two eyes, *veda* = 4, *guṇa*=3, *tri*=3, etc. (For more details about the *bhūta saṁkhyā* system see Datta and Singh 1935.) The quantity *nikharva* $= 10^{11}$. Thus, the above verse may be translated:

> Mādhava of Saṅgamagrāma spoke the approximate [*āsanna*] number of the circumference of a circle: *vibudha* [33] *netra* [2] *gaja* [8] *ahi* [8] *hutāśana* [3] *tri* [3] *guṇa* [3] *veda* [4] *bhavāraṇa* [27] *bāhavaḥ* [28], i.e., [2,827,433,388,233] is the measure of a circle of diameter *nava* [9] *nikharva* [100,000,000,000].

This corresponds to $\pi = 3.141, 592, 653, 5922\ldots$, accurate to 11 places after the decimal point, with the 12th and 13th places (92 respectively) differing slightly from their accurate value (89). The term *nikharva* continues the series, *koṭi, arbuda, abja, kharva, nikharva*, then in common use for centuries. (*Pāṭī Gaṇita*, 7. Shukla 1959.) Parts of this series coming from Vedic times are still in current use, e.g., *koṭi* = crore. (Currently, an *arbuda*, called “arab” is 100 crores, while a *kharva*, called “kharab” is 100 arabs.) The more common expression for π stated in the *kaṭapayādi* system, using reverse-sexagesimal notation, gives the radius of a circle with circumference 21600 (= number of minutes in 360°) as *Devo viśvasthalī bhṛguḥ*, corresponding (in reverse order) to 34374448 or $3437'\,44''\,48'''$, which is substantially more accurate than the 7th c. Bhāskara’s figure of $3438'$, or the 9th c. Vaṭeśvara’s figure of $3437'\,44''$.

As is clear from the above figures, these methods developed gradually over a thousand years, as the requirements of precision gradually increased. These accurate trigonometric values were needed for astronomy used for the practical needs of the calendar and navigation. The calendar was needed for successful monsoon-driven agriculture, in India, while navigation was needed for overseas trade. The whole tradition of computing 24 sine and cosine values, 3.75 degrees apart, was started a thousand years earlier by Āryabhaṭa in the 5th c. He computed 24 sine values and the value of π accurate to the first sexagesimal minute (*kalā*, about 5 decimal places). It is well known that Āryabhaṭa's value of π was quoted by al Khwārizmī, but not so well known that this was still being repeated by 16th c. European navigational theorists, such as Simon Stevin (Pannenkoek and Crone 1961).

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