Calculus without Limits

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Why is math difficult? The new answer is this. The European philosophy of mathematics was enmeshed with religion since Plato; initially banned by the church, it was later "reinterpreted". In contrast, ancient Indian mathematics was entirely practical. The two philosophies clashed when Indian mathematics (arithmetic and calculus) went to Europe; consequently Europeans had enormous difficulties in assimilating it. Eventually, the imported calculus was transformed to fit European cultural perceptions. This transformed version packaged with its inherent difficulties was reimported under colonialism. This account suggests a new way to resolve present-day learning difficulties with the calculus: revert to the practical context in which calculus originally developed. The philosophy of zeroism ensures there is no loss of rigour. This also fits splendidly with the new technology of computers, which has anyway already made the existing calculus syllabus obsolete. This combination of a new philosophy of mathematics and new technology makes calculus meaningful and so easy to learn that a five-year calculus course could be taught in five days as has now been experimentally demonstrated. It also enables one to go far beyond usual calculus courses and teach the advanced tools (such as elliptical integrals and nonlinear ordinary differential equations) needed to do realistic physics. The only casualty is the present-day "expert" mathematician who apes the West and stands to lose his/her job.

I. INTRODUCTION

Many students find maths difficult, but maths is so widely used that today even economists and biologists cannot do without it. Why is maths difficult? Recent research has produced a novel answer around which the new course on calculus without limits is designed. This course also takes into account the recent developments in computer technology which have made obsolete the existing courses on calculus.

The new research[1] traced the source of learning difficulties in mathematics, which makes those difficulties easy to eliminate.

II. TWO PHILOSOPHIES OF MATHEMATICS

In Europe, mathematics and geometry were linked to religious belief since Plato and his followers who derived maths from *mathesis*,[2] so that mathematics meant the science of learning or awakening the soul. This soul-relatedness of European mathematics dragged it into theological controversies. For example, the followers of Plato ("Neoplatonists" such as Proclus) asserted that the eternal truths of mathematics implied an eternal cosmos. This precipitated the first creationist controversy, since post-Nicene theology emphasized the doctrine of creation. Accordingly, after state and church combined in the 4th c., this notion of the soul was attacked, then cursed and banned in the Roman empire. Centuries later, after the Crusades failed to convert Muslims by force beyond Spain, the church accepted mathematics, but only after reinterpreting it as a tool to teach a "universal" means of persuasion![3]

In striking contrast, much of today's school maths (arithmetic, algebra, trigonometry, and calculus) originated in India in a purely practical and material context. The traditional Indian texts on arithmetic are full of problems of commercial and practical interest, and absolutely devoid of any attempt to relate mathematics to religious belief. (Even the *śulba sūtra*-s misleadingly called "ritual geometry" by some Western historians are entirely manuals for the practical purpose of building brick structures.)

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[1] C. K. Raju, *Cultural Foundations of Mathematics: The Nature of Mathematical Proof and the Transmission of the Calculus from India to Europe in the 16th c. CE*, Pearson Longman, 2007, PHISPC vol X.4.

[2] "This, then, is what learning ($\mu\alpha'\theta\eta\sigma\iota\zeta$ [mathesiz]) is, recollection of the eternal ideas in the soul; and this is why the study that especially brings us the recollection of these ideas is called the science concerned with learning ($\mu\alpha'\theta\eta\mu\alpha\tau\iota\kappa\eta'$ [mathematike])" Proclus, *Commentary*, Glen R. Morrow, Princeton University Press, Princeton, New Jersey, 1970, p. 38. Plato's account that "all learning is but recollection" [of knowledge the soul acquired in its previous lives] is in *Meno*[81-85].

[3] For a quick account, see C. K. Raju,"The Religious Roots of Mathematics", *Theory, Culture & Society* **23**(1–2) Jan-March 2006, Spl. Issue ed. Mike Featherstone, Couze Venn, Ryan Bishop, and John Phillips, pp. 95–97. For an account of the way in which this "theology of reason" is applied to current politics, see C. K. Raju, "Benedict's Maledicts", http://zmag.org/znet/viewArticle/3109 or the printed version in *Indian Journal of Secularism* **10**(3) (2008) pp. 79–90. For an account of how history was modified to match this reinterpretation, see C. K. Raju, "Teaching Racist History", *Indian Journal of Secularism*, 114) 2008, pp. 25-28, "Towards Equity in Mathematics Education 1: Good-Bye Euclid!" *Bhartiya Samajik Chintan*, **7**(4) (New Series) (2009) pp. 255-64, and *Is Science Western in Origin?*, Multiversity and Citizens International, Penang, 2009.

III. THE CLASH OF PHILOSOPHIES: THE MATH WARS

When Indian mathematics first reached Europe in the 10th c., there ensued a clash between these two entirely different ways of understanding mathematics, the practical Indian way, and the soul-related and metaphysical European way. This clash developed in two phases which have been called the "math wars".

The first phase, or the first math war between the 10th-15th c. concerns arithmetic. As is well known, Indian arithmetic travelled to Europe through Arabs, and particularly through the text of al Khwarizmi, and hence came to be known as algorismus or algorithms, after al Khwarizmi's latinized name. Algorismus refers to the elementary algorithms for addition, multiplication, division etc. that everyone learns in school today. At that time, arithmetic was done in Europe using the Roman abacus. The difficulty of moving from abacus to algorismus was made clear by Pope Sylvester II (author of a tome on the abacus) who failed to understand the place-value system underlying Indian algorithms. He demonstrated this lack of understanding for posterity by inscribing the "Arabic numerals" on the back of counters (jettons) used in the traditional European counting board, as if the shapes of the numerals had some magical properties attached!

Indian arithmetic was especially pushed into Europe by Florentine merchants, who recognized that it had a compelling competitive advantage over the abacus for purposes of commerce. However, this arithmetic remained so poorly understood by the general population for centuries in Europe, that the word "cipher", the then-name for zero, has come to mean a hard-to-understand code! The source of the difficulty was that the practical Indian understanding of arithmetic was incompatible with the prevalent religious understanding of mathematics in Europe (where a typical challenge problem was whether unity was a number).

IV. THE CALCULUS: ITS INDIAN ORIGIN AND TRANSMISSION TO EUROPE

In the second phase, or the second math war, starting in the 16th c., the calculus travelled to Europe. The calculus had developed in India, over a thousand year period, from the 5th to the 15th c. for accurate calculations with planetary models, needed for the calendar—the key technology for monsoon-driven agriculture. It was used to obtain very precise trigonometric values (eventually to the 9th decimal place). Precise trigonometric values were critical also for navigation. The navigational problem was then recognized as so major an economic impediment in Europe that from 1500–1760 various European governments allocated vast sums of money for its solution.

Naturally, Indian calendrical texts such as the *Kriyākramkari* containing calculus techniques, planetary models, and trigonometric values were collected, translated and despatched to Europe by Cochin-based Jesuits in the 16th c. These texts initially went to Christoph Clavius, and Tycho Brahe (and the similarity between Indian and Clavius' trigonometric values, and Indian and Tycho's planetary model is manifest), and through Tycho to Kepler, and later diffused in Europe, though the non-Christian sources were never acknowledged for fear of religious persecution at this time of the Inquisition.

V. THE SECOND MATH WAR

However, despite their undeniable practical value, the Indian calculus techniques involved infinite series which caused enormous confusion in Europe for at least another three centuries. For example, the Indian technique used an infinite series to calculate the ratio of the circumference of a circle to its diameter, a number today called π. However, Descartes wrote in his *Geometry* that a rigorous account of such ratios of curved and straight lines was "beyond the capacity of the human mind".[4] The source of the difficulties was the same: Europeans continued to view mathematics in religious terms as the "perfect" language in which God had written the book of the world, and this religious understanding blocked a proper understanding of the practical Indian techniques of dealing with infinity. Europeans then thought that an infinite series could only be correctly summed by carrying out the infinite task of summing an infinite number of terms. This was the source of Descartes' objection (to Fermat and Pascal), repeated by Galileo (to his student Cavalieri), to which Newton thought he had found a solution by making time metaphysical![5] Newton's purported solution was unacceptable to his contemporaries like Berkeley, and Europeans continued to struggle to put the calculus on an alternative epistemological footing—a struggle which stopped only around the mid-20th c.

In present-day terms this clash of epistemologies involved two major issues.[6] First, the insistence that only metaphysics could lead to certain truth (hence that mathematics must necessarily be metaphysical, and must avoid the empirical), and secondly that Western metaphysics had to be universal. Insistence on both these points was politically convenient (indeed essential) to the Western clergy.

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[4] R. Descartes, *The Geometry* (trans. David Eugene and Marcia L. Latham), Encyclopaedia Britannica, Chicago, 1996, book 2, p. 544.

[5] See "Time, What is it That it can be Measured?", *Science and Education*, 15(6) (2006) pp. 537-551. Also, "Time, Physics, History", extended abstract of a talk, at `http://ckraju.net/papers/Le-Temps-la-Physique-et-le-Histoire.pdf`

[6] C. K. Raju, "Computers, Mathematics Education and the Alternative Epistemology of the Calculus in the *Yuktibhāṣā*" Philosophy East and West 51(3), 2001, pp. 325-62; available from `http:\\ckraju.net\papers\Hawaii.pdf`

Ironically, under colonial influence, India re-imported the confused European viewpoint about mathematics and calculus into its educational system. On the principle that phylogeny is ontogeny a growing organism retraces the historical process of evolution of life: a child emerges from water, then crawls and finally stands up. Analogously, classroom teaching typically retraces the historical development of the subject. Thus, the thousand years of European confusion about Indian mathematics are replayed in fast-forward mode in the mind of the child being taught mathematics today. This confusion arose because mathematics which originated in one setting, was sought to be absorbed in an entirely different setting. This is what makes maths difficult today.

VI. TEACHING CALCULUS WITH LIMITS: SOME PROBLEMS

Let us spell out this difficulty in the case of the calculus as currently taught. Currently, school (K-12) students are taught calculus as a bunch of rules[7] to manipulate the derivative and integral, but are not taught the prevalent definitions of these symbols. Why not? Since that definition first requires the concept of limit. The concept of limit is introduced in a naive way, in current NCERT school texts, since it is believed that it cannot be taught in a "rigorous" way to school children. Why not? Since Western mathematicians believe that requires the formal real number system $\mathbb{R}$, which requires Dedekind cuts or Cauchy sequences, and that is regarded as too hard to teach school children.

For the same reason, school children cannot even be taught the definition of the exponential function e^x, since that requires additionally the definition of uniform convergence etc., which again is regarded as too difficult to teach in school. Thus, while any school child can promptly rattle off the rule that $\frac{d}{dx}(e^x) = e^x$, this is mere parroting for s/he cannot tell you what the derivative$(\frac{d}{dx})$ means, nor what the exponential function (e^x) means.

Mathematics being the basis of science, it is ironical that mathematics is today taught as a bunch of rules which must be parroted on authority, and without understanding. Naturally, children who are inquisitive, and not willing to blindly accept authority, find all this meaningless symbol-manipulation hard to accept, and reject mathematics altogether.

In fact, one can go a step further, beyond school. The formal real numbers $\mathbb{R}$ are taught in courses in advanced calculus[8] or topology and mathematical analysis.[9] (though only to the very few students who specialise in mathematics at the graduate level). Historically speaking, this construction of formal real numbers by Dedekind used Cantor's set theory, which was then regarded as logically suspect and full of paradoxes. From a historical perspective, Dedekind's achievement was psychological: he transferred doubts about numbers to doubts about sets!

These paradoxes of set theory arose since set theory postulates metaphysical ways to perform supertasks or an infinite series of tasks (even *without* transfinite induction principles, such as axiom of choice). Though those paradoxes of set theory are *believed* to have been resolved in the development of axiomatic set theory in the 1930's, this is just a belief, for that axiomatic set theory[10] (as distinct from naive set theory[11]) is something that even most professional mathematicians never learn, even after their PhD.! (Consequently, most mathematicians are unaware that even to believe the conjectured consistency of set theory requires double standards of proof: a standard of proof for metamathematics which differs from the proof used in mathematics, in disallowing supertasks.[12]) This is a strange situation where the validity of a procedure is maintained by limiting the questions that can be asked! Historically speaking then the key achievement of axiomatic set theory is again psychological: the doubts about set theory have just been pushed into the domain of metamathematics which most mathematicians don't know or care about!

Nevertheless, present-day mathematics begins with set theory, and, since a set cannot be so easily defined, there are NCERT school texts which start off by asserting that "a set is a collection of objects"—another piece of nonsense which students parrot. Naturally, once again, an inquisitive or sensitive mind could easily revolt at being forced to learn a subject taught in such a cloudy manner.

What a terrible irony that mathematics, once regarded as "the science of learning", has been reduced to such a mess which is so difficult to learn or teach!

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[7] e.g. H. Flanders, R. Korfhage and J. Price, *Calculus*, Academic Press, New York, 1970.

[8] e.g. D. V. Widder, *Advanced Calculus,* 2nd ed., Prentice Hall, New Delhi, 1999.

[9] e.g. W. Rudin, *Principles of Mathematical Analysis*, McGraw Hill, New York, 1964.

[10] e.g. L. Mendelson, *Introduction to Mathematical Logic,* van Nostrand Reinhold, New York, 1964.

[11] e.g. P. R. Halmos, *Naive Set Theory*, East-West Press, New Delhi, 1972.

[12] The formal argument runs as follows. By Gödel's theorem if set theory is decidable it is inconsistent. By Gödel's other theorem, if set theory is consistent without the axiom of choice then it must be consistent with axiom of choice. On the other hand, if one were allowed to use such a transfinite induction principle, such as Zorn's Lemma or Hausdorff Maximality Principle (equivalent to the Axiom of Choice) in metamathematics, set theory could easily be made decidable.

VII. CALCULUS WITHOUT LIMITS: THE SOLUTION

On the theory developed above, these learning difficulties with mathematics can be resolved by reverting to the original (practical) context in which that mathematics historically developed. For calculus this means a move towards Āryabhaṭa's idea of shifting from clumsy geometric techniques of calculation to the elegant technique of numerically solving ordinary differential equations. Practically speaking, this idea fits in marvelously well with the current technology of computers, and would anyway be the preferred way to teach mathematics today.

Those trained in Western formal mathematics might object that numerical techniques are not "rigorous", but are approximate and erroneous (because in their understanding truth can only be metaphysical, and that metaphysics must be Western). Of course, even they would be forced to admit that for ALL practical applications of mathematics (without exception), such as sending a man to the moon, one has no option but to perform numerical calculations. Nevertheless, they would insist that the only "rigorous" way to do mathematics is to do it metaphysically. And that for this we must follow their ritual of first postulating the metaphysical ability to manipulate infinite sets, and then believe in the coherence of those rules purely on faith (in the consistency of set theory) and double standards (to maintain that consistency). These rituals and beliefs have been challenged for a decade, and there has been no serious counter-response, so it is time to forget about them and move on.

Now the historical fact is that the infinite series developed in India and was used in practical ways by Indian mathematicians *without* postulating the ability to perform supertasks or an infinite series of tasks. This achievement has been brushed aside on the same grounds that the Indian infinite series lacked rigorous proof. In fact there was a *different* method of proof, and when appropriately reconstructed, the process used does involve a new philosophical paradigm for mathematics, which I have called **zeroism**. This realistic philosophy[13] accepts mathematics as fallible, and such practical computations as all that we will ever have, AND REJECTS PLATONIC IDEALISM AS ERRONEOUS. To repeat, this stands on its head the usual belief of formal mathematicians that formal idealisations are valid, and that practical numerical computations are forever erroneous. In fact, on zeroism, it is the formal idealisations which are erroneous, for they can never correspond to anything real. As a side benefit, the semantically void syntactic manipulation of symbols required by formalism is a task best left to machines designed for this purpose (the computer).

Incidentally, computer technology has also made completely obsolete the ability to mindlessly manipulate integrals and derivatives that Indian school children are forced to learn in the name of learning calculus. Thus, with the easy availability of symbolic manipulation programs like MACSYMA (now open source) or MATHEMATICA it is easy to solve, in a fraction of a second, the toughest symbolic manipulation problems in any calculus text.

The advantage of this new approach is that it makes the calculus shockingly easy. For example, the exponential function is defined as the solution of $y'(x) = y(x)$ with $y(0) = 1$. Arithmetically, the derivative is now just the difference quotient which arises naturally in the process of interpolating in a table of values, using the elementary arithmetic rule of three. Geometrically, the difference quotient corresponds to the slope of the chord (not tangent, which requires limits). The new point of zeroism is that uniqueness is not required for any practical purpose, so that we can happily live with the non-uniqueness of the chord. In real life, we neglect the changes that take place in ourselves and our friends from day to day (and indeed from instant to instant), and we write down the number π only to a certain number of decimal places, the exact number of places being decided by the practical context. Zeroism accepts at the philosophical level this mundane ability to handle non-uniqueness.

This new approach also makes it very easy to teach "advanced" topics, such as elliptic integrals, even to school children.[14] The students can either write the required computer programs themselves, or use my software CALCODE, which accepts symbolic input to define a differential equation, and provides both numerical output and 3-D visualisation of solutions using Open-GL. CALCODE also has features to analyse the solutions in various ways. (They can even do the calculations by hand.) All this means that a five year calculus course could be taught in five days to a group of students. Actual experiments along these lines have been conducted and the results reported.[15] Apart from elliptic integrals, students go further and do nonlinear differential equations in a natural way that is required to meaningfully teach Newtonian physics.

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[13] *Cultural Foundations of Mathematics*, cited above. See also, C. K. Raju "Zeroism and Calculus without Limits", paper presented at the 4th Nalanda Dialogue between Buddhism and Science, `http://ckraju.net/papers/Zeroism-and-calculus-without-limits.pdf`.

[14] For an actual project along these lines, see `http://ckraju.net/11picsoftime/pendulum.pdf`.

[15] "Calculus without limits: Report of an experiment", paper presented at the 2nd People's Education Congress, HBCSE, Mumbai, 5–9 Oct 2009. Available at `http://ckraju.net/papers/Calculus-without-limits-presentation.pdf`

VIII. THE OBSTACLE

The only serious obstacle is this: post-colonisation, many Indians have come to regard education as a process of learning how to ape the West. This difficulty is particularly acute with mathematics with regard to which many non-mathematicians have an inferiority complex because of the learning difficulties they encountered. Hence most people are unwilling to speak up. Policies are made by powerful people who do not themselves know mathematics. Indeed, they have no personal knowledge of even *who* the real experts and knowledgeable people are, but place their confidence in people purely on the strength of certificates issued by the West. No such 'expert" Western-approved Indian mathematician has ever been known to have engaged seriously with the philosophy of mathematics (and philosophers have stayed away from it as already noted), so they absolutely avoid engaging in any process of public debate to prevent their ignorance from being exposed. Mathematics has become a process of blind obedience to rituals dictated by authority originating in the West (which is what students are actually taught).

Finally, there is the material motive: apart from the relationship they share with the West, which is often a major source of income, these Western-approved "experts" would lose their jobs if the way of teaching mathematics changed fundamentally.

Hence it needs to be emphasized that though self-serving, the advice of these "experts" is unconstitutional: for if mathematics is theorem-proving, as made out in the West, then the theorems of mathematics would change with Buddhist or Jain logic. Why, then, teach only one sort of theorems? Religiously biased mathematical theorems should obviously not be taught in a secular state like India.

Thus, the above proposal to teach calculus without limits, apart from its solid practical advantages, and the elimination of learning difficulties related to mathematics, also has the advantage that it advocates a shift away from a religiously biased form of mathematics towards a more practical and secular form.