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The Indian Origins of the Calculus and its Transmission to Europe Prior to Newton and Leibniz. Part I: Series Expansions and the Computation of π in India from Āryabhaṭa to *Yuktidīpikā*[*]

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Extended Abstract. The background and *pramāṇa* of the high-order 'Taylor' series and 'Gregory-Leibniz' series expansions used in 14th–15th c. CE in India have never been fully and clearly explained. The details are as follows.[1]

Āryabhaṭa in *Gaṇita* 11 (499 CE) cursorily dismisses the geometric method of computing sine values, 'using triangles and quadrilaterals'. He goes on to state (*Gaṇita* 12) a finite difference method of computing sines, which is exactly like an Euler solver. (It should not be presumed that Euler derived this method independently, since Euler not only wrote an article on the use of the sidereal year in Indian astronomy, but diligently followed up the work of Fermat, whose challenge problem to European mathematicians is a solved example in Bhaskara II.) While the actual values in *Gītikā* 10/12 are values of the first sine differences, *Gaṇita* 12 applies to the second sine differences (as also noticed by Delambre), allowing us to compute both sine values and differences. Āryabhaṭa's computational notion of a function as a stored table of values/differences, along with a method of (linear) interpolation, has epistemological advantages over the formal set theoretic definition, involving supertasks. Second differences are used by Brahmagupta (628 CE) (*Khaṇḍakhādyaka* II.2.4), for quadratic interpolation, while Vaṭeṣvara (904 CE)(*Siddhānta* II.2.64–67) uses further 'Stirling's' formula for quadratic interpolation, along with trigonometric values that are only 56′ apart, to achieve a higher precision to the second [sexagesimal minute].

[*]Draft: Please do not quote without the author's permission.

[1]C. K. Raju, *Cultural Foundations of Mathematics: The nature of mathematical proof and the transmission of the calculus from India to Europe in the 16th c. CE.*, PHISPC, Vol X. (4), New Delhi 2005 (to appear).

Bhaskara II justifies this method using the notion of 'instantaneous sine difference', closely related to his notion of 'instantaneous velocity of a planet'.

This background combined with the fraction series expansion of Brahmagupta (*Brāhma Sphuṭa Siddhānta* 12.67) leads very naturally to the power series expansion for the sine function credited in 1501 CE by Nīlakaṇṭha to his predecessor Madhava (ca. 1340 CE), and also found in the *TantrasangrahaVyākhyā/Yuktidīpikā* (2.241–243), *Kriyākramakarī and Yuktibhāṣā*, etc. While the power series expansion was probably long known, since Govindasvāmin (9th c. CE) attempted to derive trigonometric values accurate to the third sexagesimal minute, and a definition of the sum of an *infinite* geometric series, as stated by Nīlakaṇṭha, was probably also available from much earlier, there was a difficulty in summing the series. This could be done only after the expression for the *vārasamkalitā* given by Nārāyaṇa Paṇḍita in his *Gaṇita Kaumudi*, to sum the intermediate series $\frac{1}{n^{k+1}}\sum_{i=1}^{n} i^k$, for the non-elementary cases, $k \geq 4$. (Fermat and Pascal's derivation of the area under 'higher-order' parabolas similarly used higher-order figurate numbers.)

Āryabhaṭa's value of π, accurate to five decimal places, was probably derived geometrically, as I have fully explained earlier, by continuing the (ca. -500 CE) *śulba-sūtra* method (e.g. Apastamba 3.2) of cutting the corners of a square, but using, instead of the *śulba-sūtra* value of $\sqrt{2}$, a full algorithm for square-root extraction, stated in *Gaṇita* 4. (Square-root extraction was unknown to the earliest-known (late 11th c. CE) Arabic text of the *al Majest*—a clearly composite work, ambitiously and uncritically attributed in its entirety to a 'Ptolemy', which work further mentions 'the difficulty with fractions', and multiplication.) This octagon-doubling method differs from the 13th–14th c. CE hexagon-doubling method used by both al Kashi, and Yu-Chhin, but attributed to 'Archimedes', and the 3rd c. CE Liu Hui respectively.

For the 11th–12th order 'Taylor' polynomials, computation of Madhava's coefficients, accurate to the third minute, required a value of π accurate to at least 8 places, and the value of π, attributed to Madhava by Nīlakaṇṭha in the *ĀryabhatīyaBhāṣya*, is accurate to 10 places after the decimal point. This value of π was derived by accelerating the convergence of various series. I explain the notation for polynomials and rational functions used in the *Kriyākramakarī*. I also explain how the the notion of order of growth of the variable (*rāsī*), as distinct from the constant (*rūpa*) was used to obtain the *samskāra* correction, and to compute its grossness (*sthaulya*) in a very simple way that both Youskevich, and Hayashi et al have missed. This computation explicitly resulted in the continued fraction expansion for π (related to the expansions used by Brouncker and Wallis).

I further explain the traditional need for accurate trigonometric values in the context of an accurate calendar, critical to Indian agriculture, which calendar was traditionally computed for the meridian of Ujjayini, and had to be recalibrated across the length and breadth of India, through local latitude and longitude corrections. Longitude was traditionally determined in India, as stated by Bhaskara I (629 CE), using an accurate measure of the radius of the earth—also obtained through accurate trigonometric values, and angle measurement. (Despite the many prizes instituted by various European governments until 1711, for a method of determining longitude, this method of longitude determination could not be used by European navigators since such an accurate measure of the size of the earth was *not* available in Europe even a thousand years later between Columbus and Picard (1672), leading to the Portuguese ban in 1500 on the carrying of globes aboard ships.) Āryabhaṭa's value of the length of the year was an order of magnitude better than the 'Ptolemaic' value used in the contemporary Julian calendar, and just prior to the Gregorian calendar reform of 1582, Jesuits, like Matteo Ricci, were searching in Cochin for 'an intelligent Brahmin or an honest Moor' to explain Indian methods of timekeeping. The importance of the calendar for India is illustrated by my observations, last year, about how the use of the Gregorian calendar in preference to the traditional calendar, led to the mistiming of agricultural operations, in India, and widespread panic about drought, in July 2003, repeated in July 2004, as evidenced by newspaper headlines and government plans.

Finally, I also explain the construction of the navigational instrument (the *rāpalagai*, consisting of two pieces of wood and string) used by the Indian navigator, Kanha, who navigated Vasco da Gama from Africa to India, and how this instrument uses harmonic interpolation to measure angles with a theoretical accuracy of $11'$, and thence compute both latitude and longitude at sea. The instrument was obtained during a field visit to the Lakshadweep islands.