5-day Course on Calculus without Limits: the Theory

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Extended abstracts of lectures

I

CURRENT PEDAGOGY OF CALCULUS: A CRITIQUE

Every calculus student learns that $\frac{d}{dx}e^x = e^x$ but not the definition of either the symbol $\frac{d}{dx}$(which requires a knowledge of limits) or the symbol e^x (which requires a knowledge of uniform convergence etc.) The difficulty that students have with the calculus is not due to bad teaching, but due to difficulties inherent in the subject itself. Limits require $\mathbb{R}$ (constructed using Dedekind cuts or equivalence classes of Cauchy sequences) which is already too difficult to teach in school. Ironically, what the student typically takes away is visual intuition, the rejection of which (in *Elements* 1.1.) led to Dedekind cuts. The construction of $\mathbb{R}$, further, requires axiomatic set theory, avoided even by most mathematicians who tend to rely on naive set theory. Since the subject cannot be explained to a student, it ends up teaching subordination to authority.

II

A CRITIQUE OF FORMAL MATHEMATICS.
PART 1: AXIOMS AND DEFINITIONS

Why teach calculus with limits? The justification for teaching calculus is that it is useful for physics and engineering, but the justification for limits is that they make calculus "rigorous"? But do they? I consider four examples.

(a) Limits require supertasks which require set theory. But can supertasks (e.g. transfinite induction principles, such as the axiom of choice) be admitted also in metamathematics to determine the consistency of axiomatic set theory?

(b) Why $\mathbb{R}$? The stock answer is completeness. But why not something *larger*? If $\mathbb{S}$ is a proper ordered-field extension of $\mathbb{R}$, then infinities and infinitesimals would exist in $\mathbb{S}$. (This is different from Non-standard analysis, where infinities and infinitesimals enter only at an intermediate stage.) Why not work with such an $\mathbb{S}$?

(c) How *exactly* should the derivative and integral be defined for purposes of applications to physics? The requirements of physics forced a change in the definition of derivative and integral to the Schwartz derivative (or Mikusinski's operators etc.) and the Lebesgue integral. Classically, a differentiable function must be continuous; with the Schwartz derivative, a (Lebesgue) integrable function is differentiable, so the two definitions do not agree, for example, in applications to differential equations.

(d) The problem of infinities in the S-matrix expansion of quantum field theory can be related to the problem of defining products and convolutions of Schwartz distributions. There are many differing definitions, and comparison theorems apply to only a few of them. So which definition should one should one accept? Mathematical authority, today, seems to side with the simple-minded Colombeau product which is basically useless for physics. Should the choice be decided by mathematical authority or by applicability to physics?

These examples show that the claim of rigor hides the arbitrariness in the choice of definitions and axioms.

III
A CRITIQUE OF FORMAL MATHEMATICS. PART 2: THE CHOICE OF LOGIC UNDERLYING PROOF

A good mathematician is supposed to give proofs, and is not supposed to waste time in asking: of what *use* is a proof? But what use is it?

The common story is that a mathematical proof makes a proposition absolutely sure. However, proof varies with logic: a proof using 2-valued logic would not be a proof with 3-valued logic or with quasi truth-functional logic. How can one be *sure* which logic to use? Buddhist and Jain logic are not 2-valued, nor is quantum logic. (I will outline my formal proof connecting a quasi truth-functional logic to the Hilbert-space axioms for quantum mechanics.) Thus, while a formalist cannot appeal to empirical facts to justify the choice of logic, even such an appeal will not necessarily sustain 2-valued logic.

If both axioms and proof are arbitrary, then in teaching formal mathematics what we are really teaching is subservience to mathematical authority (which lies in the West). If that be the case, the claim of rigor in formal mathematics is ultimately only a claim of Western supremacy!

IV
A CRITIQUE OF CURRENT HISTORY OF MATHEMATICS. PART 1: EUCLID AND ALL THAT

This claim is furthered by the common story that only Western mathematics had proof, and that this notion of proof began with the Greeks, and specifically Euclid. In fact, the *Elements* first came into Europe, during the Crusades, as one of the books in a captured Arabic library at Toledo which was mass-translated into Latin. At that time of extreme religious fanaticism, there was a sense of shame in learning from Arabs, so the story was given out that all Arabic knowledge was actually of Greek origin, and hence Europeans were its rightful inheritors. In fact, "Euclid" is not mentioned (as the author or otherwise) in any Greek manuscript of the *Elements*, and there is no evidence he existed.

On the contrary, there is ample evidence that mathematics was related to religious belief, as its very name "mathematics" (from *mathesis*) shows. In his *Republic*, Plato prescribed the teaching of mathematics explicitly for the good of the soul, to make the student virtuous. The religious character of that mathematics is further clear from the fact that this notion of soul was banned by the church at the same time that Proclus (a key commentator on the *Elements*) was declared a heretic.

After the failure of several Crusades, post-Crusade church theologians sought other ways to convert Muslims, who did not accept proof based on the Christian scriptures. Hence the church needed a universal means of "irrefragable demonstration". Christian theology was transformed, and the *Elements* was reinterpreted and absorbed into it. Islamic rational theology (*aql-i-kalām*) was also absorbed, with modifications, into Christian rational theology.

Proclus' idea of proof was different. He admitted the empirical at the beginning of mathematics, for he thought that mathematics was a mixed type of being. However, influenced by church thinking, Hilbert and even Russell ultimately eliminated the empirical to make mathematics pure metaphysics.

The attempt by the church to belittle all non-Christians carried over into racist and colonial thinking, and inventions like "Euclid" continue to serve a similar purpose today. Strangely, non-Westerners blindly and submissively accept this Western mathematical philosophy (and the accompanying false history), without understanding it or critically examining it, or debating it, and even force their children to learn it compulsorily.

V
A CRITIQUE OF THE CURRENT HISTORY OF MATHEMATICS. PART 2: THE REAL HISTORY OF THE CALCULUS

Difficulties of present-day teaching of the calculus arise since it is similarly based on false history, and attributed to Newton and Leibniz. In fact, the calculus developed in India, over a thousand year period, and was transmitted to Europe in the 16th c. CE by Jesuits based in Cochin. At this time due to the Inquisition and other forms of religious persecution, non-Christian sources were not acknowledged by Europeans. The first Europeans who received these texts (Christoph Clavius, Tycho Brahe, Kepler, Galileo, Cavalieri, etc.) had difficulty in understanding the Indian methods of handling infinite series, just as Europeans earlier had difficulties with the algorismus texts containing elementary arithmetic. Responding to Fermat and Pascal, Descartes wrote that the length of a curved line was beyond the capacity of the human mind. On the principle that phylogeny is ontogeny it is these European difficulties in understanding the calculus that are played out in "fast forward" mode during calculus teaching in the classroom today. Going back to the real historical sources of the calculus also makes for better pedagogy.

VI
THE ALTERNATIVE PEDAGOGY OF THE CALCULUS

Historically, measuring curved lines was easy: empirical methods were admitted in mathematics, and a flexible string was used to measure lengths. This, in turn, allowed an angle to be defined as the length of an arc (rather than as "something" "enclosed" by a pair of straight lines). A function (starting off with the sine and cosine of angles) was, historically, defined as a table together with an interpolation procedure (i.e., not just as an abstract rule set-theoretically asserted to exist, but as an implementable rule, as in present-day computers). Linear interpolation led Āryabhaṭa naturally to the difference quotient. The problem of the non-uniqueness of the difference quotient is readily handled using the philosophy of zeroism, similar to the thinking that went into the Indian method of defining the sums of infinite series. All this leads naturally to the calculus as the study of the numerical solution of ordinary differential equations. This is adequate for many practical applications of the calculus, and, in fact, makes it possible to go far beyond what is done today. Moreover, it is easy to understand e^x defined as the solution of the differential equation $f'(x) = f(x)$, with $f(0) = 1$. Students can write their own code or use the computer package CALCODE to calculate the solutions of a variety of ordinary differential equations. There is no loss of capability in terms of symbolic computation, for which they can use open-source packages such as MACSYMA. This extends naturally to several variables.

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