Calculus without Limits: the Theory

Current pedagogy of the calculus: a critique

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Outline

The size of calculus texts

- Typical early calculus texts (e.g., Thomas[1], Stewart[2]) today have over 1300 pages in large pages (and small type).

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- At the end what does the student learn?

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The difficulty of defining limits

- Surprisingly little!

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- Understanding a simple calculus statement

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$$\frac{d}{dx}\sin(x) = \cos(x),$$

- needs a definition of $\frac{d}{dx}$.
- However, Indian NCERT class XI text says:

> *First, we give an intuitive idea of derivative (without actually defining it). Then we give a naive definition of limit and study some algebra of limits*[3]

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- $$\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$
- $\lim_{h\to 0}$ is formally defined as follows.

  $$\lim_{x\to a} g(x) = l$$

  if and only if $\forall\, \epsilon > 0,\ \exists\, \delta > 0$ such that

  $$0 < |x - a| < \epsilon \Rightarrow |g(x) - l| < \delta, \quad \forall x \in \mathbb{R}.$$

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- But the definitions given are not precise.
- They have the ϵ’s and δ’s.
- But are missing one key element: $\mathbb{R}$.

The formal reals

Dedekind cuts

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- Formal reals $\mathbb{R}$ often built using Dedekind cuts.
- Set theory provides a model for formal natural numbers $\mathbb{N}$, which provide a model for Peano arithmetic.
- $\mathbb{N}$ can be extended to the integers $\mathbb{Z}$.
- This integral domain $\mathbb{Z}$ can be embedded in a field of rationals $\mathbb{Q}$.

Dedekind cuts

contd.

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- May be readily shown that the cuts form an ordered field, viz., $\mathbb{R}$.
- Called “cuts” since Dedekind’s intuitive idea originated from *Elements* 1.1.

Elements 1.1

The fish figure

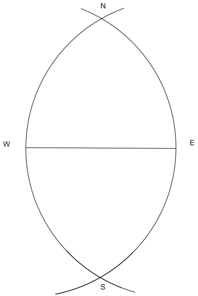

Figure: **The fish figure.**

- With W as centre and WE as radius two arcs are drawn, and they intersect at N and S.

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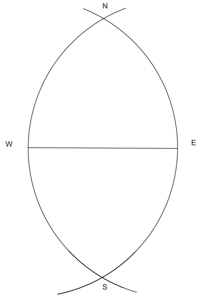

Figure: **The fish figure.**

- With W as centre and WE as radius two arcs are drawn, and they intersect at N and S.
- Used in India to construct a perpendicular bisector to the EW line and thus determine NS.

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- If arcs are drawn in $\mathbb{Q} \times \mathbb{Q}$ they may "pass through" each other, without there being any (exact) point at which they intersect,
- since there may be "gaps" in the arcs, corresponding to the "gaps" in rational numbers.

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- Formally, corresponds to the least upper bound (lub) property of $\mathbb{R}$: if $A \subset \mathbb{R}$ is bounded above, then it has a lub in $\mathbb{R}$ (that is, $\exists m \in \mathbb{R}$ such that $a \leq m, \ \forall a \in A$ and if $a \leq m_1, \ \forall a \in A$ then $m \leq m_1$.

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- And other similar properties.

Cauchy sequences

Alternative construction of $\mathbb{R}$

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- The decimal expansion of a real number is an example of such a Cauchy sequence.
- For $xin\mathbb{R}$, $x \notin \mathbb{Q}$, the decimal expansion neither terminates nor recurs.

Alternative construction of $\mathbb{R}$

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- Let $\{a_n\}$, $\{b_n\}$ be Cauchy sequences. We say $\{a_n\} \sim \{b_n\}$, if $a_n - b_n \to 0$. That is, $\forall \epsilon > 0, \ \exists N$ such that $|a_n - b_n| < \epsilon, \ \forall n > N$.

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- ~ is an equivalence relation, and we define +, . , and < in $\mathbb{Q}/\sim$ in the obvious ways to get $\mathbb{R}$
- $\mathbb{R}$ is complete: every Cauchy sequence in $\mathbb{R}$ converges.
- This is equivalent to the lub property of $\mathbb{R}$.

Imitating the European experience

- Teaching $\mathbb{R}$ is regarded as too complicated and is postponed to texts on advanced calculus[4] or mathematical analysis.[5]

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- (Cauchy 1789-1857, Dedekind 1831-1916)

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- What the student learns about set theory is
- What the student typically learns is something as follows.

  “A set is a collection of objects”

  or

  “A set is a well-defined collection of objects”

Differing ideas of sets

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- The management was even more unhappy.

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- (They had not heard of Bourbaki, so I resigned!)

What set theory the student learns

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- Most make do with naive set theory.[6]

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- Here the set $P = \{x_0, x_1, x_2, \ldots x_n\}$ is a partition of the interval $[a, b]$, and $t_i \in [x_i,\ x_{i-1}]$.

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- the two key techniques of (symbolic) integration are
  - integration by parts (inverse of Lebiniz rule), and
  - integration by substitution (inverse of chain rule)
- since integration techniques are more difficult to learn than differentiation techniques.

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- However, the student does not learn the definitions of $\sin(x)$ etc.

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- However, the student does not learn the definitions of $\sin(x)$ etc.
- since the definition of transcendental functions involve infinite series and notions of uniform convergence.
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$
- The student hence cannot define e^x, and thinks $\sin(x)$ relates to triangles.

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- The trick is to make the concepts and rules seem intuitively plausible
- by appealing to visual (geometric) intuition, or physical intuition etc.

What the student takes away

contd

- Thus, apart from a bunch of rules, the student carries away the following images:

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- The student is unable to relate the images to the rules.
- Ironically, the whole point of teaching limits is the belief that such visual intuition may be deceptive.

Belief that visual intuition may deceive

- Recall that Dedekind cuts were motivated by the doubt that the “fish figure” (Elements 1.1) is deceptive.

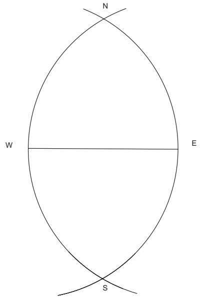

Figure: **The fish figure.**

Figure: Dedekind’s doubt was that the two arcs which visually seem to intersect need not intersect since there are gaps in $\mathbb{Q}$.

More misconceptions

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- They say the derivative is the slope of the tangent line to a curve.
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- When pressed they see that a tangent line may meet a curve at more than one point.
- But are unable to offer a different definition of the tangent.

Misconceptions about rates of change

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- But consider Popper's argument.

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- Hence there must be an optimum value of Δt neither too large nor too small.
- This is quite different from taking limits, and not at all what calculus texts have in mind.

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- This job of symbolic manipulation can be done more easily by symbolic manipulation programs running on low-cost computers.
- Why teach human minds to think like low-cost machines?

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contd.

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contd.

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Why teach limits?

contd.

- Thus, what the student learns in a calculus course (manipulating unclearly defined symbols)
- is a skill which has become obsolete.
- it is today a useless skill.
- However, teaching the student to obey rules he does not understand
- teaches blind obedience to mathematical authority (which lies in the West).

Why teach limits?
contd.

- This sort of teaching started during colonialism.

Why teach limits?

contd.

- This sort of teaching started during colonialism.
- But why should it continue today in a free society?

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   - The definition of functions, such as e^x.
   - How to correlate the derivative with the calculation of rates of change useful in physics.

Conclusions

contd.

2. The present-day course does teach how to manipulate the symbols $\frac{d}{dx}$, and $\int$ without knowing their definition. This is a task which can be easily performed on a low-cost computer, using freely available programs.

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contd.

2. The present-day course does teach how to manipulate the symbols $\frac{d}{dx}$, and $\int$ without knowing their definition. This is a task which can be easily performed on a low-cost computer, using freely available programs.
3. By forcing a student to learn a subject without proper understanding, the present-day calculus course, also teaches a student subordination to mathematical authority (which lies in the West).