Calculus without Limits: the Theory

A Critique of the History of Mathematics
The New Pedagogy

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Introduction

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- to modern computing technology.

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- This length can be measured with a string, finger measurements, a kamal, a quadrant etc.
- It can be measured in degrees (units of circumference) or radians (units of radius).

Measuring angles with the kamal

Figure: Measuring angles with the kamal

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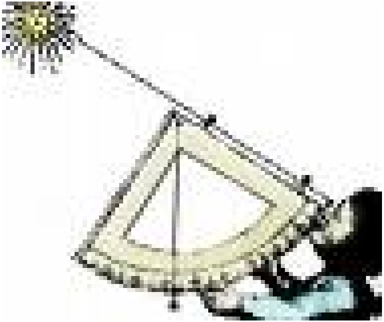
- Punch a hole in the centre and pass a plumb line through it.
- You can then measure angles with the vertical with it.



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- $f \subseteq A \times B$ is called a function if
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- (together with an interpolation procedure; this is always available as we will see).

Stock sine table

- Here is the usual table of sine values.

x°	$\sin(x)$
0	0
15	$\frac{\sqrt{6}-\sqrt{2}}{4}$
30	$\frac{1}{2}$
45	$\frac{1}{\sqrt{2}}$
60	$\frac{\sqrt{3}}{2}$
75	$\frac{\sqrt{6}+\sqrt{2}}{4}$
90	1

Modified sine table

- let us first rewrite the sine table as follows

x	$\sin(x)$
0	0
0.2617	0.2588
0.5235	0.5
0.7853	0.7071
1.0471	0.8660
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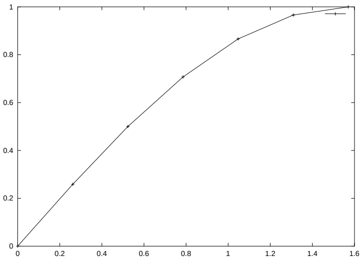
- Simply join the points with straight lines.



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- We also run into this if we use the elementary arithmetic rule of 3
- or similar triangles.

Linear interpolation: continued

- From the graph for any x value

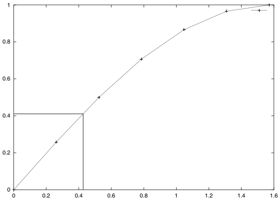
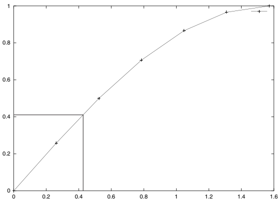


Figure: Interpolating sine values graphically

Linear interpolation: continued

- From the graph for any x value
- we can read off the corresponding y value.



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- We can instead work numerically.
- Method 2: To calculate $\sin(x)$ for a given x.
- In the table we first locate x_1 and x_2 between which x lies.

- To refer to the entries in the table, let us rewrite the table as follows.

	x	$\sin(x)$	
x_1	0	0	y_1
x_2	0.2617	0.2588	y_2
x_3	0.5235	0.5	y_3
x_4	0.7853	0.7071	y_4
x_5	1.0471	0.8660	y_5
x_6	1.3089	0.9659	y_6
x_7	1.5707	1	y_7

Slope

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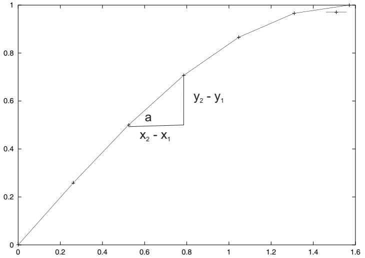

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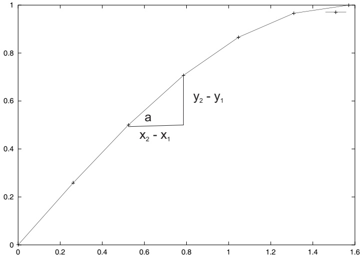

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- If x lies between x_1 and x_2, say.
- Let $y_1 = \sin(x_1)$, and $y_2 = \sin(x_2)$
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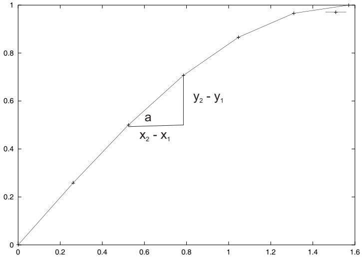
$$\Delta y = y_2 - y_1, \qquad \Delta x = x_2 - x_1$$



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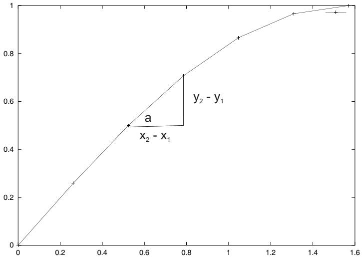
Slope (continued)

- The quantity

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \tan a$$



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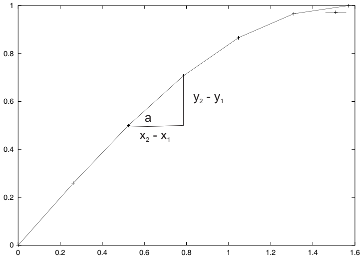
Slope (continued)

- The quantity
$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \tan a$$
- is just the slope of the chord joining the point $(x_1, \ y_1)$ to $(x_2, \ y_2)$.



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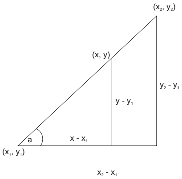

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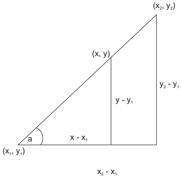
- From the figure it is clear that

$$\frac{y - y_1}{x - x_1} = \tan a = \text{slope}$$

- or

$$y - y_1 = (x - x_1) \times \text{slope}$$



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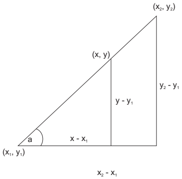

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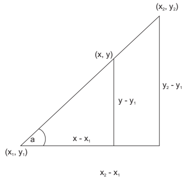

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- Since x lies between x_1 and x_2.
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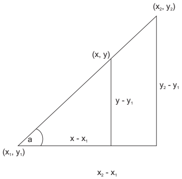
- $$y - y_1 = (x - x_1) \times \frac{\Delta y}{\Delta x}$$



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$$y - y_1 = (x - x_1) \times \frac{\Delta y}{\Delta x}$$
- we can immediately calculate the desired y value:
$$y = y_1 + (x - x_1) \times \frac{\Delta y}{\Delta x}$$

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$$y - y_1 = \frac{\Delta y}{\Delta x}(x - x_1)$$

$$\text{change} = \text{unit rate of change} \times \text{distance}$$

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- Hence,
  $\sin 1^\circ = \sin 0.01745 = \frac{0.2588}{0.2617} \times 0.01745 = 0.01725$
- We can compare this with the value of $\sin 1^\circ$ from a calculator, which comes out to be 0.01745.

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- So, to get more accurate sine values, we must have more values in our table.
- The earth is round, but because we see only a small part of it, it appears flat.
- 8th c. mathematician Lalla: "Mathematicians say $\frac{1}{100}$th part of the earth is flat."

Difference quotient vs derivative

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- (However, a more accurate interpolation is usually obtained by taking higher derivatives.)

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- Similarly, $(y_4 - y_3) + (y_3 - y_2) + (y_2 - y_1) = y_4 - y_1$, and so on

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- Here Σ is a Greek letter used for the "S" of "Sum" (just as the Greek letter Δ was used for the "D" of "Difference").

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- that summation ($\int$) is the inverse of the difference (d).

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- How to determine the value of the function?
- If we are dealing with finite differences, it is very easy to calculate the answer by summing successive differences.
- In the implicit case, $y'(x) = f(x, y)$ we use the elementary technique today called an "Euler" solver.

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- students should learn how to apply Newton's laws of motion.

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- Note: the operative term is calculate not prove.
- If the object to send a rocket to the moon, what is required is the ability to calculate the solution, and not prove its existence and uniqueness.

Fundamental theory of calculus
contd

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- Numerical solution of an ODE is a superior substitute for the fundamental theorem of calculus.
- From a practical point of view there is no doubt that this is a better approach, and it also allows the solution (and visualisation) of a wide variety of problems.

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- This approach gives a rigorous account of not only $\sin(x)$ and $\cos(x)$
- but also extends to various non-elementary functions, such as a the Jacobian elliptic functions which arise in applications (to simple pendulum).

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- which are influenced by European theological beliefs about infinity.

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- So also, from a realistic perspective, “insignificant quantities” can be discarded
- as they are discarded in any numerical calculation done on a computer.
- In fact, this is true for any representation of anything in the real.

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- Buddhists, for example, reject the Western answer as erroneous.

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- Simple: correct it, and find a better answer.
- The ϵ–δ definition of derivative was not good enough, it had to be corrected.
- Likewise instead of seeking fake guarantees of certainty, let us look at practical value (at least at the level of school and undergraduate math which is compulsory).

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- Mechanical or general rule for it to be avoided, as in philosophy of zeroism.

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- Hasn't there been some loss of ability to write down calculus formulae.
- and evaluate integrals by hand?
- No, since this task can be easily done by open-source programs like MACSYMA, or MAXIMA.
- A student takes two days to learn how to use it.

Conclusions

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- It enables the student to focus on practical applications, and also solve problems earlier regarded as too difficult to do.
- It enables clarity of concepts
  - Difference quotient arises naturally by rule of 3.
  - Fundamental theorem of calculus is obvious.
  - All elementary, and many non-elementary functions are easily defined.

The experiment

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The experiment

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- Age group 22-55 years. Included faculty and head of department.
- Background: Students admitted after 8th std. Some have monastic education.
- Very poor performance in pre-test even on elementary arithmetic.

Pre-test
Calculus without Limits

Notes:

1. This is NOT a competitition. The aim of this test is only to provide *feedback* regarding your current knowledge of mathematics.
2. Some questions may be beyond your current knowledge. Please don't be anxious about it. It is expected that you do *not* know the answers to all questions, and those questions are there only to establish the limits of your knowledge.

I : Arithmetic

1. Find $124 + 568$.
2. Find $532 - 319$.
3. Calculate 3542×213.
4. If 2184 is divided by 17 what is the quotient and what is the remainder?
5. Which is the greatest among the following four numbers: $\frac{5}{8}, \frac{2}{3}. \frac{3}{7}, \frac{4}{10}$?
6. Write $\frac{3}{4}$ as a decimal.
7. Write 0.4352 as a proper fraction.
8. What is the square of 23?
9. A trader bought an item for Rs 26 and sold it for Rs 38. What percentage profit did he make?
10. The Rajdhani express travels from Delhi to Mumbai in 18 hours and 30 minutes with stops of 10 minutes each at Kota, Ratlam and Baroda. If its average speed is 81 km, what is the distance from Delhi to Mumbai?
11. If 3 kg of flour sells for Rs 32 how much does 5 kg of flour sell for?

II : Alegbera

12. If $x = 5$ what is the value of x^2?
13. If $2x + 3 = 10$ what is the value of x?
14. If $2x + 3y = 40$ and $x = 7$ what is the value of y ?
15. If $x^2 - x - 6 = 0$ what are the possible values of x?

III : Geometry

16. If one angle of a right-angled triangle is $30°$ write the other two angles in degrees.
17. A rectangle has length 1 and width 2. What is the length of it diagonal?
18. Give an approximate figure for the circumference of a circle whose radius is 1.
19. Plot a straight line through the points $(2,\ 3)$ and $(2,\ -3)$.

IV : Elementary Calculus

20. What is $\frac{d}{dx}\sin(x)$?
21. What is $\int\ x^2 dx$?

V: Calculus questions from question bank

22. Differentiate $\sqrt{\frac{\sec x - 1}{\sec x + 1}}$ with respect to x.
23. Differentiate $\log \frac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}+x}$ with respect to x.
24. Evaluate the integral $\int \frac{x^2+1}{x^4+1}\ dx$.
25. Evaluate the integral $\int x^2 \tan^{-1} x\ dx$.

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- And also solve ordinary differential equations.

Post-test
Calculus without Limits

I : Elementary computations

1. Convert 30 deg to radians.
2. Convert 2 radians into degrees.

II : Elementary Calculus

3. What is $\frac{d}{dx}\sec(x)$?
4. Evaluate $\int \cos(3x+1)dx$
5. Find the second derivative of $x \sin x$.
6. Find
$$\int_0^1 xe^x dx$$
7. Numerically integrate
$$\int_0^{0.5} \frac{1}{\sqrt{1-x^2}\sqrt{1-x}}\, dx$$
.

III: Questions from question bank (differentiation)

Differentiate the following functions with respect to x.

8. $\sqrt{1-x^2}$.
9. $x^2 e^{\sqrt{x}}$.
10. $x^2 \sin^3 x \ \cos^4 x$

(continued from page 1: differentiate the following with respect to x)

11.
$$\log \sqrt{\frac{1 + x\cos x}{1 - x\cos x}}$$

12.
$$\tan^{-} 1 \left(\frac{e^{2x} + 1}{e^{2x} - 1}\right)$$

IV: Questions from question bank (integration)

Evaluate the following integrals.

13.
$$\int \frac{1}{1 - x^2}\, dx$$

14.
$$\int \frac{1}{x^3 + x^2 + x + 1}\, dx$$

15.
$$\int \frac{\sqrt{2} - \sqrt{x}}{1 - \sqrt{2x}}\, dx$$

16.
$$\int \sec^{-1}\sqrt{x}\, dx$$

17.
$$\int \cot^5 x\, dx$$

V : Ordinary differential equations

18. Solve the differential equation $y' = 2y$, with $y(0) = 1$ and hence find $y(4)$.
19. Solve the differential equation $y' = x\sin(x)$ with $y(0) = 1$ and find the value of $y(10)$.
20. Solve the differential equation $y'' = -3y$ with $y(0) = 1$ and $y'(0) = 0$, and find the value of $y(20)$.

Results

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- In the post-test this student got nearly 100%. About half the class managed to clear the test.
- The bottom half of the class performed poorly.
- As clear from the pre-test some of the students (and faculty) did not fulfil the starting criterion of knowing school math at 8th std. level. They are being given remedial coaching in school math.


Central University of Tibetan Studies
Workshop on "Calculus without limits"
22nd - 28th September, 2009