Calculus: the real story

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Roman arithmetic

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- Roman and early Greek way of representing numerals is inferior: requires 12 symbols instead of 4.
- Challenge: can you write 10^{53} in Roman numerals?

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- Names of 53 places ($10^{53} =$ *tallakshana*) in Buddhist *Lalitavistara sutta*.
- Numbers such as 10^{29} in Jaina literature.

इमा मेऽअग्नऽइष्टका धेनवः सन्त्वेका च दश च दश च शतञ्च शतञ्च सहस्रञ्च सहस्रञ्चायुतञ्चायुतञ्च नियुतञ्च नियुतञ्च प्रयुतञ्चार्बुदञ्च न्यर्बुदञ्च समुद्रश्च मध्यञ्चान्तश्च परार्धश्चैता मेऽअग्नऽइष्टका धेनवः सन्त्वमुत्रामुष्मिँल्लोके २

Shukla Yajurveda 17.2

एका च दश च दश च शतं च शतं च सहस्त्रं च सहस्त्रं च अयुतं च अयुतं च नियुतं च नियुतं च प्रयुतं च अर्बुदं च न्यर्बुदं च समुद्रः च मध्यं च अन्त्यः च परार्धः च

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- followed by Śridhar (*Pātigaṇita*), Mahavira (*Gaṇitasārasangrah*), Bhaskara (*Lilāvatī*).

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- Why did Arabs and Europeans import Indian arithmetic?

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- Simpler question: How do you add XXIX and LXXVIII?

The Roman abacus

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- Whew!

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- by Crusading and racist historians based on very late texts (in another language from another place).

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- and used to impose colonial domination and exploitation (Macaulay).
- (Not true that history written by victors. This false history written by military losers fetched a huge empire.)
- Exposing falsehoods of Western history important to end racist and colonial inequity today.

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- then via Florentine merchants (13th. c.)

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- but the story emerges from the very terms "Arabic numerals", and "zero".

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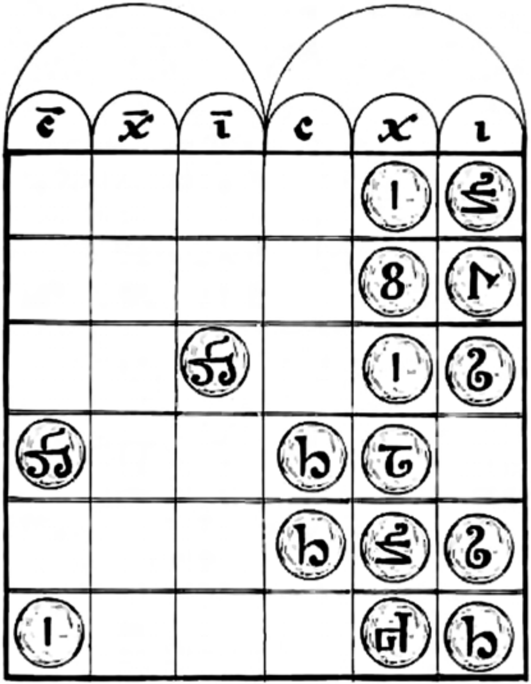
- (That mistake immortalised in the term "Arabic numerals".)



13

87

4 019

400 520

539

100 065

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- Roman numerals are additive, like counters: XII means $X + I + I = 12$.
- Place value numerals are not additive: $120 \neq 1 + 2 + 0 = 3$.

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- Hence, the Florentine law that contracts (cheques) must be written out in words.

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- Thus confusion about Indian arithmetic persisted for six centuries.

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- Europeans misunderstood that arithmetic in various ways, and that confusion persisted for over six centuries.

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- The word "trigonometry" indicates a conceptual misunderstanding: the concepts relate to a circle (chord, *jya*) not a triangle.
- Correct term is circlemetry. These concepts discussed in chapter on the circle in early Indian texts.

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- Why not? (Brings in essential properties of the circle.)

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- Just an absurd Western religious delusion that non-empirical (metaphysical) knowledge is somehow superior, perfect, eternal, and infallible.

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- This value of π repeated, a thousand years later, in 16th c. Europe by Simon Stevin.

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- Hence, my software CALCODE.

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- He solved them using only linear interpolation or the rule of 3.
- (Today wrongly called "Euler's method" of solving ordinary differential equations, after Euler who, like other Westerners, never acknowledged his Indian (non-Christian) sources.)

Sine differences, not sine values

- Important thing is that Āryabhaṭa in Gaṇita 10 Āryabhaṭa stated sine differences, later called *khanḍa-jyā*

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- Differences 3.75° apart.
- Differences can be directly used to interpolate
- or calculate values at end points of intervals.

Āryabhaṭa's table of sine differences

- मखि भखि फखि धखि णखि ञखि
  ङखि हस्झ स्ककि किष्ग श्घकि किघ्व ।
  घ्लकि किग्र हक्य धकि किच
  स्ग श्झ ङ्व क्ल प्त फ छ कलार्धज्या ॥१२॥

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- The nine vowels अ, इ, उ, ऋ, ऌ, ए, ओ, ऐ, औ denote the two nines of zeros (corresponding to the 18 places from 10^0 to 10^{17}): each vowel takes one *varga* and one *avarga* place.

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- Thus अ denotes the place of 1 as well as 10, इ denotes the place of 100 as well as 1000, etc.

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- E.g., ख्युघृ = 4,320,000, since ख् = 2, य् = 30 , so that ख्यु = 320,000, while घ् = 4, so that घृ = 4,000,000.

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- It is also order-independent: could write above as घृख्यु.

Translation

- 225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, 7—[these are the] Rsine [differences] [for the quadrant divided into as many equal parts, each part hence being 225′] [in] minutes.

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- (Circumference of the circle in minutes is $360 \times 60 = 21,600$.)

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not algebraic equation

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- Translation: (12) The Rsine of the first arc divided by itself and negated gives the second Rsine difference. That same first Rsine, when it divides successive Rsines gives the remaining [Rsine differences].

Mathematical translation

- R_i = sine values, $\delta_i = R_i - R_{i-1}$ sine differences. Then Āryabhaṭa's rule consists of two parts

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- Note 2: Brahmagupta also uses 2nd differences for quadratic interpolation.

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- E.g. for $n = 23$, $\delta_{23} = 22$, $\delta_{24} = 7$, while $R_1 = 225$, so that we should have $R_{23} = (\delta_{23} - \delta_{24}) \times R_1 = 15 \times 225 = 3735 \neq 3431$ the 23rd sine value.

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- E.g. for $n = 23$, $\delta_{23} = 22$, $\delta_{24} = 7$, while $R_1 = 225$, so that we should have $R_{23} = (\delta_{23} - \delta_{24}) \times R_1 = 15 \times 225 = 3735 \neq 3431$ the 23rd sine value.
- Difference in each case, since no value is a multiple of 225.

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- by extending Āryabhaṭa's recursive process to infinite series.

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- 7th c. Bhāskara I's figure is $3438'$,
- 9th c. Vaṭeśvara's figure is $3437'\,44''$.
- Later stated as *Devo viśvasthalī bhṛguḥ*, corresponding (in reverse order) to 34374448 or $3437'\,44''\,48'''$,

Madhava's value of π

- Value of π also stated in older *bhūta saṁkhyā* system in Nīlakaṇṭha's *ĀryabhaṭīyaBhaṣya*
- सङ्गमग्रामजो माधवः पुनरत्यासन्नां परिधिसंख्यामुक्तवान् –
  विबुधनेत्रगजाहिहुताशनत्रिगुणवेदभवारणबाहवः ।
  नवनिखर्वमिते वृतिविस्तरे परिधिमानमिदं जगदुर्बुधाः॥
- Mādhava of Saṅgamagrāma spoke the approximate [*āsanna*] number of the circumference of a circle: *vibudha* [33] *netra* [2] *gaja* [8] *ahi* [8] *hutāśana* [3] *tri* [3] *guṇa* [3] *veda* [4] *bhavāraṇa* [27] *bāhavaḥ* [28], i.e., [2,827,433,388,233] is the measure of a circle of diameter *nava* [9] *nikharva* [100,000,000,000].

Value of π contd.

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- accurate to 11 decimal places
- with the 12th and 13th places (92 respectively) differing slightly from their accurate value (89).

Madhava's sine table

श्रेष्ठं नाम वरिष्ठानां हिमाद्रिर्वेदभावनः।
तपनो भानुसूक्तज्ञो मध्यमं विद्धि दोहनम्॥
धिगाज्यो नाशनं कष्टं छन्नभोगाशयाम्बिका।
म्रिगाहारो नरेशोऽयं वीरो रणजयोत्सुकः॥
...
छायालयो गजो नीलो निर्मलो नास्ति सत्कुले।
रात्रौ दर्पणमभ्राङ्गं नागस्तुङ्गनखो बली॥
धीरो युवा कथालोलः पूज्यो नारीजनैर्भगः।
कन्यागारे नागवल्ली देवो विश्वस्थली भृगुः॥
तत्परादिकलान्तास्तु महाज्या माधवोदिताः।
स्वस्वपूर्वविशुद्धे तु शिष्टास्तत्खण्डमौर्विकाः॥ २.९.५॥

Madhava's sine table

Table : Mādhava's sine values

No.	Kaṭapayādi	kalā ($'$)	vikalā($''$)	tatparā($'''$)
1	श्रेष्ठं नाम वरिष्ठानां	224	50	22
2	हिमाद्रिर्वेदभावनः	448	42	58
3	तपनो भानुसूक्तज्ञो	670	40	16
4	मध्यमं विद्धि दोहनम्	889	45	15
...	...	...	...	...
21	धीरो युवा कथालोलः	3371	41	29
22	पूज्यो नारीजनैर्भगः	3408	20	11
23	कन्यागारे नागवल्ली	3430	23	11
24	देवो विश्वस्थली भृगुः	3437	44	48

Accuracy of Madhava's sine values

Table : Accuracy of Mādhava's sine table.

No.	Mādhava's sine value	Difference
1	0.0654031452	0.0000000160
2	0.1305262297	0.0000000375
3	0.1950903240	0.0000000020
4	0.2588190035	−0.0000000416
...	...	...
...	...	...
21	0.9807852980	0.0000000176
22	0.9914448967	0.0000000353
23	0.9978589819	0.0000000587
24	1.0000000000	0.0000000000

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- overseas trade (needs good navigation)
- agriculture (needs a good calendar to tell the rainy season).
- (Note: Gregorian calendar is a bad, unscientific, religious calendar which continues to ruin our economic interests to this day: see video and presentation: "A tale of two calendars".)

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- Also needed to to determine the size of the earth, and
- use that to solve longitude triangles from knowledge of latitude difference and departures (Maha Bhaskariya, II.3–4), (of Bhaskara I, 7th c.)

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- Recorded that the pilot was telling the distance by his teeth!

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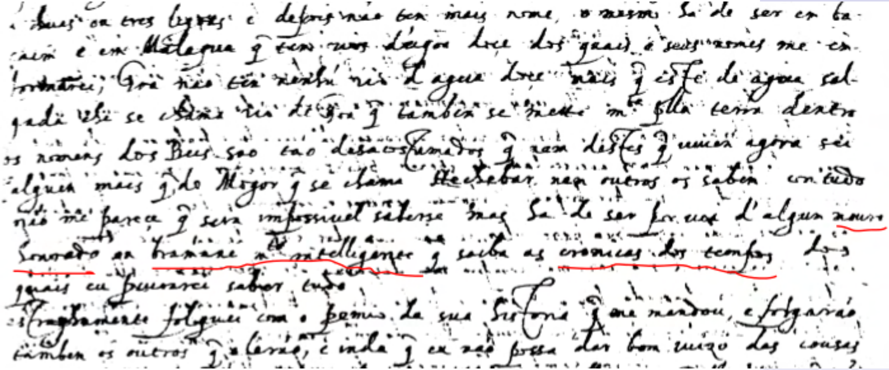
- also used Indian length of tropical year for Gregorian calendar reform of 1582 (not based on fresh observation, not immediately accepted by Protestants)

Ricci's letter of 1581

- In 1581 Ricci was in Cochin and wrote that he was looking for "an honorable Moor or an intelligent Brahmin to tell him about Indian methods of timekeeping".



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- Hence, Europe could not solve the longitude problem this way (used inaccurate "Dead reckoning").

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- Royal Society, French Royal Academy were set up for this purpose.

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- Finite geometric series known in India since Yajurveda, as already shown.
- Infinite geometric series had appeared in India by the 14th-15th c.

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- The sum of an infinite [*anantya*] series, whose later terms (after the first) are got by dividing the preceding one by the same divisor everywhere, is equal to the first term [*a*] multiplied by the common divisor [*d*], and divided by one less than the common divisor.

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- Sum of infinite (*anantya*) geometric series stated by Nīlakaṇṭha (in *Āryabhaṭīyabhāṣya*, *Gaṇita* 17).
- एवं यस्तुल्यच्छेदपरभागपरम्पराया अनन्ताया अपि संयोगः तस्यानन्तानामपि कल्प्यमान्स्य योगस्याद्यावयविनः परम्परांशच्छेदादेकोनच्छेदांशसाम्यं सर्वत्रापि समानमेव।
- The sum of an infinite [*anantya*] series, whose later terms (after the first) are got by dividing the preceding one by the same divisor everywhere, is equal to the first term [a] multiplied by the common divisor [d], and divided by one less than the common divisor.
- $a + \frac{a}{d} + \frac{a}{d^2} + \cdots = \frac{ad}{d-1}$.
  (Assuming $d > 1$, so common ratio less than 1.)

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त्रिशरादिविषमसंख्याभक्तमृणां स्वं प्रिथक् क्रमात् कुर्यात् ॥ २.२७१ ॥

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- Translation: To the diameter multiplied by 4 alternately add and subtract in order the diameter multiplied by 4 and divided separately by the odd numbers 3, 5, etc.

"Leibniz" series
contd.

- Mathematical translation: if d is the diameter of the circle, then

$$\text{circumference} = 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \cdots. \quad (3)$$

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- What baffled all Western thinkers (Descartes, Galileo, Newton, Berkeley . . .) was this: how to do this infinite sum “perfectly”.

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- See my article: "Towards equity in math education. 2: The Indian rope trick" (google, CKR, rope trick)

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The perils of perfection

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- He thought for this we must physically sum the series term by term, which would take an infinite amount of time.

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A mere religious belief

- But why is math perfect and eternal knowledge?

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- These are religious beliefs which are not universal or compelling.

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- (See, e.g., KL-paper on “Islam and science”, google).

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- Deciding logic culturally means truths of math are mere cultural truths.
- And if nature of logic itself is decided by experience then deductive proofs are more fallible than inductive proofs.

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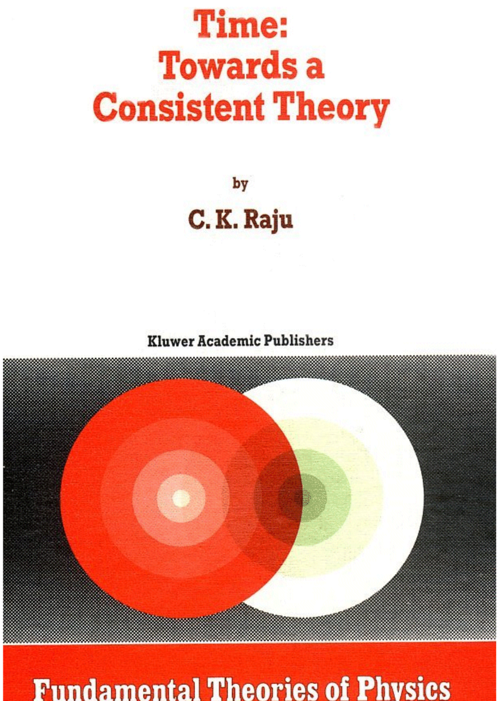
- Newtonian physics failed because he made the notion of time metaphysical.


Time:
Towards a
Consistent Theory
by
C. K. Raju
Kluwer Academic Publishers
Fundamental Theories of Physics

Does calculus require metaphysics?

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- However, belief persists that calculus requires metaphysical "real" numbers $\mathbb{R}$for existence of limits.
- That belief is taught in school and undergraduate calculus courses today (without actually teaching about $\mathbb{R}$ except to math majors!).

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- which have a different arithmetic (no associative "law", etc.)

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- Similarly issues in renormalization problem of quantum field theory.

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- $1 + 2 + 3 + 4 + \cdots \stackrel{?}{=} \frac{-1}{12}$ (Ramanujam sum).

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- in *Yuktidīipikā* to accelerate convergence of slowly convergent series, like "Leibniz" series.

PEARSON LONGMAN
History of Science, Philosophy and Culture in Indian Civilization
General Editor D. P. Chattopadhyaya
Volume X Part 4
Cultural Foundations of Mathematics
The Nature of Mathematical Proof and the Transmission of the Calculus from India to Europe in the 16th c. CE
C. K. RAJU
PHISPC
Centre for Studies in Civilizations

Contemporary changes

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- (See the series of expository articles in *Physics Education*, India)

Math pedagogy

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- On the principle that phylogeny is ontogeny
- these difficulties in the classroom are a replay of European difficulties with imported Indian math.
- Solution is to go back to the way that math developed in India, and reject the way it was misunderstood in Europe.

5-day course on calculus

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- can be taught in a mere five days as I have demonstrated with 8 groups in 5 universities in 3 countries.


Central University of Tibetan Studies
Workshop on "Calculus without limits"
22nd - 28th September, 2009


نشست علمی "ریاضیات از منظری دیگر"، پروفسور سی.کی. راجو
مرکز مطالعات و همکاری‌های علمی بین‌المللی، تهران، ۱۳۹۱

The big picture

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  - and is still taught today.

Notorious claims of “superiority”

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- claims of racist "superiority",
- and claims of Christian "superiority" (as in US Supreme Court judgment: that native Indians in America lost their right to land after being discovered by Christians)

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- The time has come to be critical and to discard it,
- to focus on the practical value of math,
- and go back to our roots.

References

- 1. Reference in abstract posted at http://ckraju.net/papers/Calculus-story-abstract.html

References

- 1. Reference in abstract posted at http://ckraju.net/papers/Calculus-story-abstract.html
- 2. A longer reading list and list of videos is posted at http://ckraju.net/papers/Reading-list-Bengaluru.html