Āryabhaṭa's sine table

- मखि भखि फखि धखि णखि ञखि
  ङखि हस्झ स्ककि किष्ग श्घकि किघ्व ।
  घ्लकि किग्र हक्य धकि किच
  स्ग श्झ ङ्व क्ल प्त फ छ कलार्धज्या ॥१२॥

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- The nine vowels अ, इ, उ, ऋ, ऌ, ए, ओ, ऐ, औ denote the two nines of zeros (corresponding to the 18 places from 10^0 to 10^{17}): each vowel takes one *varga* and one *avarga* place.

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- Thus अ denotes the place of 1 as well as 10, इ denotes the place of 100 as well as 1000, etc.

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- It is also order-independent: could write above as घृखु.

Translation

- 225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, 7—[these are the] Rsine [differences] [for the quadrant divided into as many equal parts, each part hence being 225′] [in] minutes.

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- (Circumference of the circle in minutes is $360 \times 60 = 21,600$.)

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- (Brahmagupta tried to go back to older 6 sine values 15° apart using quadratic interpolation, with second differences.)

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- Translation: (12) The Rsine of the first arc divided by itself and negated gives the second Rsine difference. That same first Rsine, when it divides successive Rsines gives the remaining [Rsine differences].

Mathematical translation

- $R_i =$ sine values, $\delta_i = R_i - R_{i-1}$ sine differences. Then Āryabhaṭa's rule consists of two parts

$$\delta_2 - \delta_1 = -\frac{R_1}{R_1}, \tag{8}$$

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- Note 2: Brahmagupta also uses 2nd differences for quadratic interpolation.

Nīlakaṇṭha's correction

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- while Āryabhaṭa is precise to the minute (so that $\delta_1 - \delta_2 = 225 - 224 = 1$).

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- E.g. for $n = 23$, $\delta_{23} = 22$, $\delta_{24} = 7$, while $R_1 = 225$, so that we should have $R_{23} = (\delta_{23} - \delta_{24}) \times R_1 = 15 \times 225 = 3735 \neq 3431$ the 23rd sine value in the *Sūrya Siddhānta* (or *Āryabhaṭīya*).

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- Difference in each case, since no value is a multiple of 225.

Āryabhaṭa and Euler

- Thus, Āryabhaṭa's rule for calculating sine values is a recursive process ($n \geq 2$):

$$R_n = R_{n-1} + \delta_n \tag{11}$$

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- Strikingly similar to "Euler's method" of solving an ordinary differential equation.
- which corresponds (for $y'' = -y$) exactly to Nīlakaṇṭha's formula.