Calculus without Limits: the Theory

A Critique of Formal Mathematics Part 1: Axioms and Definitions

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Outline

Introduction

Why are limits important?

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- limits are taught for rigor.

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- But how far is it true?

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- Today I look at arbitrariness in axioms and definitions,
- to demonstrate the arbitrariness in calculus from within formal mathematics.

Set theory and supertasks

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- This is a supertask.

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- The fastest computers today can manage teraflops, or around 10^{12} floating point additions per second.
- If we use this computer exclusively to add continuously for a year: we can only go up to 10^{20} additions—still a long way from infinity.

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- But set theory allows us to do it metaphsyically.
- In fact, set theory allows us to specify an uncountable infinity of real numbers!

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- which has a very old history, dating back to times when European culture had not even begun.
- Such approximations are readily possible
- the question of a supertask arises only when we speak of being able to specify the value of π exactly or uniquely.

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Dedekind's real achievement

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- which were earlier attached to the calculus,
- got pushed into doubts about set theory.
- From this perspective, Dedekind's real achievement was that he pushed doubts about supertasks and infinity away from nubers and into the domain of set theory.

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- they can say (as Paul Erdos nearly said), "so many people believe it, they can't all be wrong can they?" (proof by numbers"), etc.
- (For more details about such proofs, see the appendix to my book The Eleven Pictures of Time, Sage, 2003.)

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- and into set theory.
- But were they resolved?

Paradoxes of set theory

Traditional paradoxes of infinity

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- The second line says "if you remove the whole from the whole, what remains is the whole".
- It was such paradoxes which made Descartes and Galileo suspect the calculus when it first arrived in Europe (as we will see in more detail, later on).

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- He argued that adding a day to eternity would not change eternity. Hence, the world was not eternal.
- Curiously, he had a different attitude towards the eternal torture in hell which he thought awaited non-Christians, a torture which he thought they would experience for an eternity of time.

The double standard

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- Let us try.

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- On the other hand, if $R \in R$ then, again, by definition, we must have $R \notin R$.
- So, either way, we have a contradiction.

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The definition of a set

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- Take, for example, the von-Neumann-Bernays-Gödel (NBG) set theory.
- Here, a well-formed formula (of the sort used in Russell's paradox) in general only defines a class.
- A set is defined as a class A for which $\exists$ a class B, such that $A \in B$.

Resolution of Russell's paradox in NBG

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- The paradoxes of set theory apply to classes, not sets.
- Mathematicians can stick to sets and thus avoid the paradoxes which are now (believed to be) confined to classes.

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- How can we be sure that all paradoxes of set theory are resolved.
- NBG includes classes which are paradoxical.
- How can we be sure this does not make NBG inconsistent?

The consistency of NBG

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- The consistency of NBG is not proven
- it is only widely believed among mathematicians.
- So, basing the calculus on $\mathbb{R}$ and NBG does not guarantee the surety of the results.
- That's only a belief.

Metamathematics

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- The question is: what kind of metamathematics?
- Before answering this question, let us recall some socially accepted results of metamathematics.

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- Recall that Cantor's continuum hypothesis states that if $\aleph_0$ is the cardinality of the infinite set $\mathbb{N}$ of natural numbers, and c is the cardinality of $\mathbb{R}$ then $2^{\aleph_0} = c$.

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- Not clear what happens when X is infinite.
- Recall that Cantor's continuum hypothesis states that if $\aleph_0$ is the cardinality of the infinite set $\mathbb{N}$ of natural numbers, and c is the cardinality of $\mathbb{R}$ then $2^{\aleph_0} = c$.
- The metamathematical theorems of Gödel and Cohen showed that the continuum hypothesis (CH) implies (but is not implied by) the axiom of choice.

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- Equivalent is Zorn's Lemma: in a partially ordered set if every chain is bounded above, then there must be at least one maximal element,
- or Hausdorff maximality principle: in a partially ordered set every chain is contained in a maximal chain etc.

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- the existence of a Lebesgue non-measurable set or Tychonoff's theorem (that the product of compact sets is compact) etc.
- Zorn's lemma is used to prove the Hahn-Banach theorem etc.

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- Namely, let $A, B \subset \mathbb{R}^n$, with $n \geq 3$.
- Further, let A, B be bounded and have non-empty interior.
- Then, there exist finite partitions of A, B, such that $A = \bigcup_{i=1}^{k} A_i$, $B = \bigcup_{i=1}^{k} B_i$, and each A_i is congruent (under Euclidean motions) to B_i.

Banach-Tarski Paradox
contd

- This paradox conflicts violently with geometric intuition,
- for it means that a ball in 3-dimensional space may be broken into a finite number of non-overlapping pieces,
- which may be reassembled by rotation and translation (without stretching) into *two* balls of the same volume as the original.

Figure: The Banach-Tarski Paradox. A ball in 3-dimensional space can be subdivided into a finite number of pieces which can be reassembled into two balls of identical volume, without stretching, and merely by means of rigid rotations and translations.

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- We look at the formal contrapositive:
- if set theory is inconsistent with AC, then it must be inconsistent without AC.

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- Specifically, can principles like AC and CH be admitted in metamathematics?

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- However, if such a theory is decidable, then the statement *can* be either proved or disproved within the theory.
- That is, if set theory is decidable it must be inconsistent.

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- Conjecture: Transfinite recursion (an easy consequence of AC), makes set theory decidable (hence inconsistent).

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- The only answers is from mathematical authority. So formal mathematics ultimately depends upon authority, not reason.

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- Infinite procedures (even AC) allowed for proofs in mathematics, but disallowed in metamathematics.
- This is a hypocritical social consensus among authoritative Western mathematicians. Ideally, there should be one standard of proof for both mathematics and metamathematics.

Why $\mathbb{R}$?

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- This is a Cauchy sequence: for successive terms differ only in the next decimal place,
- so the difference between the m^{th} and n^{th} term can be made less than 10^{-q} where $q = \min\{m, n\}$.

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- From the construction of $\mathbb{R}$ as the set of equivalence classes of Cauchy sequences in $\mathbb{Q}$, this does not happen in $\mathbb{R}$ which is complete.
- What happens in a field larger than $\mathbb{R}$?

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- AP characterizes $\mathbb{R}$. That is, $\mathbb{R}$ is the largest ordered field with AP.
- Consequently, if we have an ordered field $\mathbb{S} \supset \mathbb{R}$, then the AP must fail in $\mathbb{S}$.

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- Thus, $\frac{1}{x}$ corresponds to what we intuitively understand as an infinitesimally small number.

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$$|\frac{1}{n} - \frac{1}{x}| < \frac{1}{n} \leq |\frac{1}{n} - 0| < \epsilon.$$
- Note: we are here *not* talking about non-standard analysis: the infinities and infinitesimals in the field $\mathbb{S}$ do not arise merely at an intermediate stage: they are "permanent", so to say.

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- Since $\mathbb{Q}$ is a field, it is well known P must be an integral domain.

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- The formal quotient, such as $\frac{x-2}{x-3}$ is defined whenever the denominator is a non-zero polynomial, even though, as a *function*, it may be infinite (or fail to be defined) at a finite set of points (the roots of the denominator).
- To avoid quibbles concerning the form $\frac{0}{0}$, we can define two rational functions to be equivalent if they differ only on a finite set of points. (This can happen also with equivalent formal quotients, e.g. $\frac{x(x-1)}{x-1}$ and $\frac{x(x-2)}{x-2}$.)

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- That is, polynomial arithmetic, or Brahmagupta arithmetic, is non-Archimedean, unlike integer arithmetic.

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- The only answer is that conventional calculus teaching uncritically imitates the European historical experience of the calculus.

How to define the derivative?

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- So, a discontinuous function may not be differentiated.

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- Mathematicians, on the other hand, considered it as something non-mathematical and non-rigorous—a mere construct used by physicists.

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- the test function g which is assumed to be infinitely differentiable: $g \in C^\infty$.

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- This works equally well for functions of several variables, and we can write

$$\int_{\mathbb{R}^n} f'g = -\int_{\mathbb{R}^n} fg',$$

for $g \in D(\mathbb{R}^n)$ where $D(\mathbb{R}^n)$ is the space of compactly supported and infinitely differentiable functions.

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  - Here $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index, and $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \ldots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$.
  - These seminorms p_N generate a vector topology on $C^\infty(\mathbb{R}^n)$, in which the space of compactly supported test functions D is a closed subspace.

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- That was just a consensus among mathematicians, which has changed, because the earlier definition was not adequate for physics.

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- However, the Dirac delta is not the zero distribution, since $\int \delta(x)dx = 1$.
- Thus, for purposes of physics, we need to settle on one of the two as the right definition, and clearly the Schwartz definition is better than the older ϵ–δ definition.

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- Called the Schwartz product. If g is a distribution, and $f \in C^\infty$, define

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  for all test functions h, where $\langle f, h \rangle \equiv \int fh$.
- If $f \in C^\infty$ and h is a test function, $f.h$ is again a test function. Hence, the right hand side is well defined.

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- (b) is associative (that is $(fg)h = f(gh)$ for all distributions f, g, h), and
- (c) satisfies the Leibniz rule (that is $(fg)' = fg' + f'g$ for all distributions f, g).

Taub's remark

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- Since $\theta' = \delta$, this can be rewritten as
$$2\theta \cdot \delta = \delta,$$
which immediately tells us that
$$\theta \cdot \delta = \frac{1}{2} \cdot \delta.$$

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- Since

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this corresponds to

$$\theta \cdot \delta = \frac{1}{3} \cdot \delta.$$

- Comparing the above two leads to the interesting conclusion that $\frac{1}{2} = \frac{1}{3}$!

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- These products are Fourier transformed into convolution integrals, which are divergent.
- If we apply this to δ^2 we see that

$$(\delta^2\hat{)} = \hat{\delta} * \hat{\delta} = 1 * 1 = \int 1 = \infty.$$

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- Quantum field theorists use the Hahn-Banach definition useless for classical physics (shock waves).
- Mathematicians use Colombeau's product useless for physics (since it is both associative and satisfies the Leibniz rule).
- What are the principles on which the choice is to be decided?

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- Since associate law and Leibniz rule holds, it has a problem as follows.

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- The correct conditions, using (a) were given by Rankine and Hugoniot.
- With the Colombeau theory, it is not possible to discriminate between forms (a) and (b).

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  - **The choice of metamathematics is arbitrary**. Calculus with limits requires infinite procedures (spertasks), incorporated in $\mathbb{R}$ which is constructed using axiomatic set theory, such as NBG. Supertasks lead to paradoxes of set. Consistency of NBG can only be proved or disproved in metamathematics. The consistency is maintained by an arbitrary choice of metamathematics: refusing to allow in metamathematics the sort of infinite procedures for proof that are admitted in NBG.

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- **The choice of the number system underlying the calculus is arbitrary.** It is possible to do calculus more intuitively in non-Archimedean fields larger than $\mathbb{R}$.
- **The definition of the derivative is arbitrary**. The classical ϵ–δ definition of the derivative is not adequate for physics, since the derivative of discontinuous functions naturally arises in physics.

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- **The definition of the product of distributions is arbitrary** The classical definition of derivative is usually replace by the Schwartz definition which is incomplete since it does not address the issue of products of distributions. Colombeau's simplistic definition is today being promoted by mathematical authority, although it is inadequate and inappropriate for physics

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- As seen by the fate of the classical definition of derivative, ultimately mathematical definitions have to be related to practical value not mathematical authority.