Why normality?

- The extensive use of the normal distribution rests on two famous principles, known as the laws of large numbers.
- **Central limit theorem**: Let $X_1,\ X_2,\ldots,X_n$ be i.i.d.r.v.'s (i.e., random sample of size n), with mean $EX_i = \mu$, and variance $E(X_i-(EX_i))^2 = \sigma^2$.
- Let
- $S_n = X_1 + X_2 + \ldots + X_n$.
- Then, $\frac{S_n - n\mu}{\sigma\sqrt{n}} \sim N(0,1)$

Weak law of large numbers

- If the variance does not exist, but the mean does, the central limit theorem fails, and we have
- The weak law of large numbers:

$$\Pr\{|\frac{S_n}{n} - \mu| > \epsilon\} \to 0$$

- This says that the sample average converges (in probability) to the mean as the sample size becomes large.
- Convergence in probability is a weak form of convergence.

Stable distributions

- What happens if neither mean nor variance exists?
- The limit is a stable distribution.
- In this case we do not have simple formulae for the probability density.
- This forces computation.

α-stable distributions

- The characteristic function of a random variable X is $Ee^{\imath tX}$.
- The characteristic functions of the stable distributions are related to the function $e^{-|t|^\alpha}$, where $\alpha \in (0, 2]$ is called the index of stability.
- The characteristic function $\phi(t)$ of the most general stable distribution has the form:

$$\log \phi(t) \; = \; -\sigma^\alpha |t|^\alpha \{1 \; - \; \imath\beta \, \mathrm{sgn}(t) \tan \frac{\pi\alpha}{2} \} \; + \; \imath\mu t$$

Stable distributions

contd.

- We say that $X \sim S_\alpha(\sigma, \beta, \mu)$, where
- $\alpha =$ stability parameter, $0 < \alpha \leq 2$
- $\beta =$ skewness parameter, $-1 \leq \beta \leq 1$
- $\sigma =$ scale parameter, $\sigma > 0$
  (if $\alpha < 2$ the variance does not exist, hence σ cannot be called variance)
- $\mu =$ location parameter
  (if $\alpha < 1$ the mean does not exist, hence μ cannot be called the mean).
- (The formula looks a bit different for $\alpha = 1$.)

Special cases

- Let $X \sim S_\alpha\ (\ \beta,\ \ \mu,\ \ \sigma)$.
- We can try to understand the nature of the distribution by looking at various special cases.
- $\alpha\ =\ 2, \beta\ =\ 0 : S_2(0,\ \mu,\ \sigma)\ =\ \ \mathrm{N}\ (\mu,\ \sigma)$, the normal distribution.
- $\alpha\ =\ 1, \beta\ =\ 0 : S_1(0,\ \mu,\ \sigma) =$ Cauchy, with density

$$f(x)\ =\ \frac{2\sigma}{\pi((x-\mu)^2+4\sigma^2})$$

- $\alpha\ =\ \frac{1}{2}, \beta\ =\ 1\text{:}\ S_{\frac{1}{2}}(1,\ \mu,\ \sigma) =$ classical Lévy with density

$$f(x)\ =\ \left(\frac{\sigma}{2\pi}\right)^{\frac{1}{2}}\ (x-\mu)^{-\frac{3}{2}}\ \exp\left\{-\frac{\sigma}{2(x-\mu)}\right\}$$

Tail probabilities

- If $X \sim S_\alpha(\beta,\ \mu,\ \sigma)$ has density f_α, then asymptotically
- $\Pr\{X > \lambda\} = \frac{D_{\alpha\beta}\sigma^\alpha}{\lambda^\alpha}$
- $$f_\alpha(x) = O\left(\frac{1}{|x|^{1+\alpha}}\right)$$
- Compared to the normal density (which declines exponentially) this goes to zero quite slowly.
- We say these distributions are "fat tailed".

Lévy perturbed SDE

- The alternative to the normal distribution is to use stable distributions.
- The mathematical theory is horrendous, and solutions can no longer be formally proved to exist.
- Formal math essentially fails in this context.
- However, computational solutions are relatively easy.
- The package STOCHODE solves such Lévy perturbed SDEs.