Calculus without Limits: the Theory

Current pedagogy of the calculus: a critique

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Outline

Introduction

The size of calculus texts

- Typical early calculus texts (e.g., Thomas[1], Stewart[2]) today have over 1300 pages in large pages (and small type).

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- At the end what does the student learn?

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The difficulty of limits

The difficulty of defining limits

- Surprisingly little!

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$$\frac{d}{dx}\sin(x) = \cos(x),$$

- needs a definition of $\frac{d}{dx}$.
- However, Indian NCERT class XI text says:

> *First, we give an intuitive idea of derivative (without actually defining it). Then we give a naive definition of limit and study some algebra of limits*[3]

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The formal definition of limits

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- $$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$
- $\lim_{h \to 0}$ is formally defined as follows.

$$\lim_{x \to a} g(x) = l$$

if and only if $\forall\, \epsilon > 0, \ \exists\, \delta > 0$ such that

$$0 < |x - a| < \delta :\Rightarrow |g(x) - l| < \epsilon \quad \forall x \in \mathbb{R}.$$

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- But the definitions given are not precise.
- They have the ϵ's and δ's.
- But are missing one key element: $\mathbb{R}$.

The difficulty of defining $\mathbb{R}$

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- or via equivalence classes of Cauchy sequences in $\mathbb{Q}$.

Imitating the European experience

- Teaching $\mathbb{R}$ is regarded as too complicated and is postponed to texts on advanced calculus[4] or mathematical analysis.[5]

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- (Cauchy 1789-1857, Dedekind 1831-1916)

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- The construction of $\mathbb{R}$ requires set theory.
- Students are not taught the definition of a set.
- What the student typically learns is something as follows.

"A set is a collection of objects"

or

"A set is a well-defined collection of objects"

What set theory the student learns

- With such a loose definition it is not possible to escape things like Russell's paradox.

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- Most make do with naive set theory.[6]

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- Here the set $P = \{x_0, x_1, x_2, \ldots x_n\}$ is a partition of the interval $[a, b]$, and $t_i \in [x_i,\ x_{i-1}]$.

The integral
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- the two key techniques of (symbolic) integration are
  - integration by parts (inverse of Lebiniz rule), and
  - integration by substitution (inverse of chain rule)
- since integration techniques are more difficult to learn than differentiation techniques.

The difficulty of defining functions

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$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

- The student hence cannot define e^x, and thinks $\sin(x)$ relates to triangles.

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- The trick is to make the concepts and rules seem intuitively plausible
- by appealing to visual (geometric) intuition, or physical intuition etc.

What the student takes away

contd

- Thus, apart from a bunch of rules, the student carries away the following images:

$$\begin{aligned} \text{function} &= \text{graph} \\ \text{derivative} &= \text{slope of tangent to graph} \\ \text{integral} &= \text{area under the curve.} \end{aligned}$$

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- The student is unable to relate the images to the rules.
- Ironically, the whole point of teaching limits is the belief that such visual intuition may be deceptive.

Belief that visual intuition may deceive

- Recall that Dedekind cuts were motivated by the doubt that the "fish figure" (Elements 1.1) is deceptive.

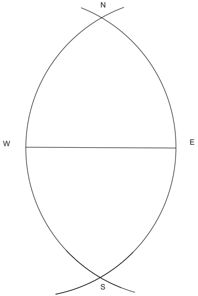

Figure: **The fish figure.**

Figure: Dedekind's doubt was that the two arcs which visually seem to intersect need not intersect since there are gaps in $\mathbb{Q}$.

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- They say the derivative is the slope of the tangent line to a curve.
- And define a tangent as a line which meets the curve at only one point.
- When pressed they see that a tangent line may meet a curve at more than one point.
- But are unable to offer a different definition of the tangent.

Misconceptions about rates of change

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- In physical terms, the derivative is usually explained as "the rate of change".
- But consider Popper's argument.

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- But choosing small Δt will increase the relative error of measurement.
- Hence there must be an optimum value of Δt neither too large nor too small.
- This is quite different from taking limits, and not at all what calculus texts have in mind.

Calculus with limits: why teach it?

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- This job of symbolic manipulation can be done more easily by symbolic manipulation programs running on low-cost computers.
- Why teach human minds to think like low-cost machines?

Why teach limits?
contd.

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contd.

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Why teach limits?
contd.

- Thus, what the student learns in a calculus course (manipulating unclearly defined symbols)
- is a skill which has become obsolete.
- it is today a useless skill.
- However, teaching the student to obey rules he does not understand
- teaches blind obedience to mathematical authority (which lies in the West).

Why teach limits?
contd.

- This sort of teaching started during colonialism.

Why teach limits?
contd.

- This sort of teaching started during colonialism.
- But why should it continue today in a free society?

Conclusions

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   - The definition of functions, such as e^x.

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   - The definition of $\mathbb{R}$ (which depends upon set theory).
   - The definition of a set (which depends upon axiomatic set theory).
   - The definition of the integral (which is defined only as an anti-derivative).
   - The definition of functions, such as e^x.
   - How to correlate the derivative with the calculation of rates of change useful in physics.

Conclusions

contd.

2. The present-day course does teach how to manipulate the symbols $\frac{d}{dx}$, and $\int$ without knowing their definition. This is a task which can be easily performed on a low-cost computer, using freely available programs.

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2. The present-day course does teach how to manipulate the symbols $\frac{d}{dx}$, and $\int$ without knowing their definition. This is a task which can be easily performed on a low-cost computer, using freely available programs.
3. By forcing a student to learn a subject without proper understanding, the present-day calculus course, also teaches a student subordination to mathematical authority (which lies in the West).

SGT University: calculus without limits

Pre-test

Answer all questions.
Blank answer fetches 0. Wrong answer gets negative marks.
Classes refer to NCERT texts. You may have learnt from a different text in school.

1. **Points.** You were taught about points in class VI.
   (a) Define a point.
   (b) What is the size of a point?
   (c) Can something with no size be seen? If something is invisible, how do you know where it is?
   (d) What is the difference between a fraction and a rational number?

2. **Numbers.** You were taught "real" numbers in class IX and class X.
   (a) Define a real number.
   (b) Write down the EXACT value of $\sqrt{2}$. If x is that exact value, show by direct calculation that $x^2=2$. (Note: this should not be 1.9999999999, but exactly 2.)
   (c) Can a complex number be written as the ratio of two integers? If i is the complex number such that $i^2=-1$ then is i irrational? Is it rational?
   (d) Are there numbers which are neither rational nor irrational? If your answer is yes, go back and re-check your definition at (a). If your answer is no, explain how -1 can have a real square root.

3. **Sets.** You were taught about sets in class X.
   (a) Define a set.
   (b) If you defined a set as a "collection of objects", define "collection" and define "object". Is "object" the same as in object-oriented programming? If not, what is the difference?
   (c) Let $R = \{ x \mid x \notin x \}$. Is it true that $R \in R$? Is it true that $R \notin R$?
   (d) Can a set have an infinite number of elements? How can you be sure?

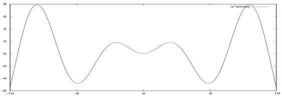
4. **Trigonometry.** You were taught about trigonometric functions in class IX.
   (a) Define sin(x).
   (b) Use that definition to calculate sin (0.3º).
   (c) Is sin (92º) defined? According to my calculator, sin (92º) = 0.9993. Is this right? Explain.
   (d) Define a radian. Exactly how many degrees is 1 radian?

5. **Limits.** You were taught about limits in class XI and XII.
   (a) Define limit.
   (b) According to my calculator $\sqrt{2}$ = 1.4142135623730950488016887242097. Does the sequence 1, 1.4, 1.41, 1.414... have a limit?
   (c) What is the infinite sum of all natural numbers, 1+2+3+4+... ? Can it be a negative number?
   (d) What is the infinite sum 1-1+1-1+1-1+...?

6. **Derivative.** You were taught about derivatives in class XI and XII.
   (a) Define the derivative of a function.
   (b) Let N denote the set of natural numbers, and let f : N → N be given by $f(x)=x^2$.

What is the derivative of f ?

(c) Define the tangent to a curve at a point. Consider the function x sin(x) whose graph is



displayed. Write the equation of the tangent to the curve at x=0. At how many points does this line intersect the curve? Can you list these points?

(d) What is the derivative of atanh(x) (hyperbolic arc tangent) with respect to x?

7. **Integral.** You were taught about integrals in class XII.
(a) Define the integral of a function.
(b) Shown below is a piece of land with irregular boundaries. How will you calculate its area?

(c) Calculate $\int_{-1}^{-2} \frac{dx}{x}$.

(d) Calculate $\int \frac{1}{\sqrt{((1-x^2)(1-4x^2))}} dx$.

8. **Applications.** You learnt about Newton's laws of motion and the simple pendulum from class VIII to class XI.
(a) At approximately what angle should you throw a cricket ball so that it travels the furthest distance?
(b) Will the answer change if you use a tennis ball instead of a cricket ball?
(c) The formula for the time period of a simple pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$. Therefore, the time period of a simple pendulum is independent of amplitude. Is this true or false?
(d) Did you ever experimentally verify any of your answers above?

Math Group: Calculus without Limits
Exam, Pre-test: A

Name: ____________________

Student Number: ____________________

Course: ____________________

Age: ____________________

Date: ____________________

- Please attach this question paper and return it along with your answer sheet.
- This is not a competitive test. The aim is to obtain feedback to decide what to teach and how.
- Since the group is heterogeneous, you may find some questions too easy, or some may be too difficult. Attempt as many questions are you are able to.

1. (a) Define a complete metric space.
   (b) The least upper bound property for $\mathbb{R}$ says that if $A \subset \mathbb{R}$ is non-empty and bounded above, then $\exists \alpha \in \mathbb{R}$ such that $a \leq \alpha$, $\forall a \in A$, and if $a \leq b$, $\forall a \in A$ then $\alpha \leq b$. Assume the least upper bound property and prove that $\mathbb{R}$ is a complete metric space.
2. (a) Define "infinite set", "countable set", "uncountable set".
   (b) Prove that $\mathbb{R}$ is uncountable.
   (c) If $\mathbb{N}$ is the set of natural numbers, and $P(\mathbb{N})$ is its power set, does there exist a bijective map $f : P(\mathbb{N}) \to \mathbb{R}$?
3. (a) Write down the binary representation of 41.
   (b) Write down the binary representation of 2.5
   (c) Rewrite your answer using a mantissa between 1 and 2.
4. Given $g(x) = \begin{cases} x^2 - C, & \text{if } x < 4 \\ -\sqrt{C}x + 20, & \text{if } x \geq 4 \end{cases}$
   (a) Find the value of C which makes g continuous on $(-\infty, \infty)$.
   (b) With the above value of C, is g differentiable? Explain your answer.
5. (a) Suppose f_n is a sequence of Riemann integrable functions which converges to the function f on $(0, \infty)$, convergence being uniform on compact subsets. Is it true that f is Riemann integrable and that $\int_0^\infty f_n \to \int_0^\infty f$?
   (b) Suppose f_n is a sequence of differentiable functions which converges uniformly to the function f on $(0, 1)$. Is it true that f is differentiable and that the sequence of derivatives ${f_n}' \to f'$?

6. (a) Give an example of a real-valued function f which is not Riemann integrable on $[0, 1]$. Is this Lebesgue integrable?

   (b) Does there exist a Riemann integrable function on $(0, \infty)$ which is not Lebesgue integrable?

7. The following ten numbers were drawn at random from $[0, 1]$ using a uniform probability distribution: 0.23, 0.74, 0.18, 0.79, 0.51, 0.34, 0.67, 0.44, 0.11, 0.44.

   (a) Find the average.

   (b) Explain why it is not 0.5.

   (c) If the average does equal 0.5 at some stage, can subsequent draws of further random numbers change that value?

   (d) An unbiased coin is tossed 100 times. The first toss is tails, and the subsequent 99 tosses are heads. At the 101st toss (i) is the probability of tails greater than that of heads or (ii) is the probability of heads greater than that of tails?

8. Suppose a monkey is typing at random on a typewriter which has 50 keys (z and Z having been dropped), and suppose that the monkey is equally likely to strike any key.

   (a) What is the probability that the first six letters the monkey types will spell the word "Hamlet".

   (b) Suppose we consider the letters typed by the monkey in consecutive blocks of six letters. What is the probability p_n that the first n blocks of six letters will have the word "Hamlet"?

   (c) Does $\lim_{n\to\infty} p_n$ exist? If so, what is it?

9. Differentiate the following with respect to x

   (a) $\sin^n x . \sin nx$

   (b) $\sec^{-1} \dfrac{\sqrt{x}+1}{\sqrt{x}-1} + \sin^{-1} \dfrac{\sqrt{x}-1}{\sqrt{x}+1}$

   (c) $x - \log\left(2e^x + 1 + \sqrt{e^{2x} + 4e^x + 1}\right)$

10. Evaluate the following indefinite integrals.

    (a) $\int \sqrt{3x+2}\ dx$

    (b) $\int \log x\ dx$

    (c) $\int \dfrac{dx}{\sqrt{\sin^3 x \cdot \sin(x+\alpha)}}$