Calculus without Limits: the Theory

A Critique of Formal Mathematics
Part 2: The Choice of Logic Underlying Proof

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- Why teach humans to behave like low-cost machines?

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- Teach calculus as a bunch of rules to be used without question—a useless skill for symbolic manipulation can be better done by low-cost machines.
- Why teach humans to behave like low-cost machines?
- Limits required since visual intuition denied. But calculus texts rely heavily on visual intuition.

Recap 2

Axioms and definitions arbitrary

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- ϵ–δ definition of derivative and Reimann integral had to be abandoned in favour of Schwartz theory and Lebesgue for applications of calculus to physics.
- However, Schwartz theory incomplete: lacks notion of product of distributions (required for physics).
- No general rule available for choosing between different definitions and axiom sets: only way is to rely on Western mathematical authority.

Universal math

- Mathematics has long been regarded in the West as universal; not merely global, but universal.

[1]Christiaan Huygens, *The Celestial Worlds Discover'd: or Conjectures Concerning the Inhabitants, Plants, and Productions of the Worlds in the Planets*, London, 1698, p. 86.

Universal math

- Mathematics has long been regarded in the West as universal; not merely global, but universal.
- This thought is captured in the following remark from Huygens[1]

> *no matter how inhabitants of other planets might differ from man in other ways, they must agree in music and geometry, since [music and geometry] are everywhere immutably the same, and always will be so.*

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- Not permitted in practice (paper with different notion of derivative not likely to be published).
- But allowed in principle that theorems may not be universal, and will vary with axioms

Proof: the key concern of formal mathematicians

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- So a formal mathematician is concerned with proof.
- A proof connects axioms to theorems.
- Today the claim is that proof is universal.
- If axioms change, theorems will change, but the connection of axioms to theorems will not change.

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- On Tarski-Wittgenstein (possible-world) semantics
- the axioms of a theory may be true in some worlds and false in others: they constitute contingent truth.
- However, a necessary truth is always true; it is true in all possible worlds.

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  - either an axiom,
  - or follows from one or more preceding A_j's by means of a rule of reasoning.

Example of rules of reasoning

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$$A \implies B$$

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$$A(x)$$

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- or the rule of instantiation

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$$\therefore A(a)$$

2-valued logic

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- assumes the definition of $\implies$ as in 2-valued logic.

$\neg p$	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$	
$p \,/\, q$	-	T F	T F	T F	T F
T	F	T F	T T	T F	T F
F	T	F F	T F	T T	F T

Table: Truth tables for 2-valued logic

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- $A \wedge \neg A \implies B$ is always true.

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- The connectives may now be defined as follows.

	$\neg p$	$p \wedge q$			$p \vee q$			$p \Rightarrow q$			$p \Leftrightarrow q$		
p / q	-	T	I	F	T	I	F	T	I	F	T	I	F
T	F	T	I	F	T	T	T	T	I	F	T	I	F
I	I	I	I	F	T	I	I	T	T	I	I	T	I
F	T	F	F	F	T	I	F	T	T	T	F	I	T

Table: **Truth table for 3-valued logic**. This table is read exactly like an ordinary truth table, except that the sentences p and q now have three values each, with I denoting "indeterminate" (and T and F denoting "true" and "false" as usual). With this system, $p \vee \neg p$ does *not* remain a tautology. A somewhat similar system was used by Reichenbach in his interpretation of quantum mechanics.

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- If A has the value I $\neg A$ has the value I, and so has $A \wedge \neg A$
- Thus proofs by contradiction fail.
- Theorems proved using 2-valued logic are not theorems with 3-valued logic.

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- Theorems vary with both choice of axioms and choice of logic.

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- Here it is possible that $A \wedge \neg A$ may be actually true.
- As in the mundane statement "A person is both good and bad".
- Definitions of connectives may be easier understood semantically.

Semantics of QTF logic

QTF logics and quantum mechanics

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- QTF logics may actually arise in quantum mechanics.
- Hilbert space axioms for quantum mechanics are based on quantum logic.
- For an account of quantum logic see my article "Quantum mechanical time" arxiv.org:0808.1344.
- For a formal proof that QTF logic implies quantum logic, see my book (appendix to chp. 6b).

Time: Towards a Consistent Theory

by

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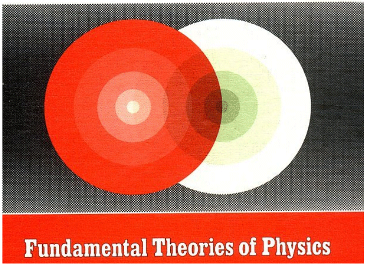
Kluwer Academic Publishers



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- Thus, the paradoxes of quantum mechanics are neatly resolved.

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- Therefore, formalism cannot appeal to the empirical to settle the question about logic.
- However, the argument about quantum mechanics shows that quantum logic is not 2-valued.
- Thus, even an appeal to the physical world does not ensure 2-valued logic.

Is logic culturally universal?

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- Is 2-valued logic culturally universal?
- No.

Buddhist logic

- The Buddha in the Brahmajāla sutta assumes a logic of four alternatives (catuśkoṭī):[2]

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- (Even natural language allows a person to be "both good and bad".)

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Jain logic of Syādavāda

Logic of 7-cases

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- Mathematics used for practical calculations is secular.
- But if mathematics is about proof using 2-valued logic
- Then it is culturally biased.

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- Calculus with limits advocated on grounds of "rigor".
- But definitions of derivative etc. are arbitrary.
- Claim of rigor rests on belief that mathematical proof is "universal".

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- Nor empirically certain.
- So proofs with 2-valued logic are not universal.
- So why teach calculus with limits?