A new way to teach calculus based on a new history and philosophy of math

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It is believed that calculus was invented by Newton and Leibniz, and the correct way to teach it is to use limits, using Dedekind's formalisation of real numbers, constructed using axiomatic set theory, based on the formalist philosophy of Russell and Hilbert which corrected the deductive method of Euclid's *Elements*.

The historical claim follows the stock pattern of Western history which attributes all achievements first to Hellenic culture (in this case, "Euclid", Archimedes, etc.), and then to post-renaissance Europe (in this case, Newton, Leibniz, etc.). Others are typically erased. The links of this history to racism are manifest from the foolish arguments of Hume, Kant,[1] and other leading Western philosophers, who justified this erasure by claiming that only white-skinned people were creative. This agenda was furthered by making "Euclid's" purported axiomatic-deductive method philosophically *normative* to declare other ways of doing things as not quite mathematics—as "land-surveying" not geometry, for example.[2] Thus, today, we have the "philosophical" belief that the only right way to do mathematics is the Western way which focuses on rigor through metaphysical deductive proof. Apart from a racist bias, this involves a *religious* bias,[3] as follows.

First, this way to do mathematics supposedly originated with a "Euclid" for whose existence there is nil evidence.[4] As for the *Elements,* its first proposition involves an empirical proof as does the key proposition 4 known as the SAS theorem. These empirical proofs were eliminated only at the turn of the 20th c. by Russell and Hilbert who disregarded these *facts*, and just went along with the *myth* of Euclid and his purported intention of metaphysical deductive proof, and hence turned SAS into a postulate.[5] However, the resulting formalisations do not fit the *Elements.* Hilbert's synthetic axiom set, which reinterprets "equal" as "congruent", clearly fails beyond proposition 35 which uses the term "equal" for incongruent areas. Birkhoff's metric axiom set makes the book *Elements*, itself, pointless, since the "Pythagorean theorem" can be proved in a single step (as in empirical Indian proofs) and does not need 46 intermediate propositions.[6] In fact, the *Elements* was written at another time (+5th c. CE) by a different person (Hypatia) with a different philosophy of mathesis which philosophy persisted from Plato[7] to Proclus,[8] and originates from the sacred geometry of Egypt.[9] This involved a belief in the soul and its past lives.[10] Though part of original Christianity,[11] those beliefs (and the related philosophy) were cursed by the post-Nicene church,[12] and became taboo in the West thereafter. Post-Crusade rational theology reinterpreted Christianity as a doctrine of reason, by adapting the Islamic theology of reason (*aql-i-kalam*), taking into account its critique by al Ghazali.[13] Theologians contended that God was bound by logic but free to create the empirical facts of his choice; hence they elevated metaphysics above empirical facts. The belief in metaphysical deduction as certain and universal actually originated with church doctrine, not the *Elements*.

Now, as a matter of commonsense, metaphysics can never be universal.[14] The belief that metaphysical (deductive) proof provides greater certainty than empirical observation is peculiar to post-Crusade Christian rational theology. Not only is it against mathesis, it rejects *all* Indian systems of philosophy at one stroke, for the one and only one thing they all accept in common is the empirically manifest (*pratyaksa*) as a sure means of proof. In fact, mathematical theorems would change with changes in logic, and logic is neither culturally universal (e.g. Buddhist logic of *catuskoti*, Jain logic of *syadvada*)[15] nor empirically certain (e.g. quantum logic,[16] or the logic of natural language), so the nature of logic underlying deduction *must* be decided empirically, to avoid a bias. This bias extends also to the postulates which are, in principle, arbitrary for a formal theory. But, for example, students are today taught that calculus requires the continuum, which is contrary to Islamic and Indian beliefs in atomicity. Further, all practical applications of calculus today use a computer which uses a discrete number system and a floating point arithmetic, different from the arithmetic of formal reals.[17] This too is declared erroneous. Thus teaching formal mathematics inculcates a pro-Christian bias, and declares non-Christian and practical ways of doing math as wrong.

These racist and religious biases inherent in the church education system were globalised by colonialism,[18,19] and are reflected in the way calculus is taught today. Limits are of no practical value and are supposedly done for the sake of rigor. In fact, however, engineering and science students of calculus are not actually taught formal limits, or formal reals or axiomatic set theory. They are only *told* about the intention of rigor, but *taught* to accept it on faith. At an advanced level, students do learn about formal limits, and formal real numbers, but neglect formal set theory. They remain unfamiliar with the paradoxes of set theory which persist even after its axiomatisation (such as the Banach-Tarski paradox), and unaware that set theory can be used to "rigorously" derive conclusions (that one ball of gold can be turned into ten) laughable from a practical perspective. What value such rigor if rigor it indeed be? These biases are supported by social control: the formal mathematics community rigorously excludes any forum, whether in the classroom or outside, where such questions about mathematics can be raised and discussed.

Present-day calculus lacks conceptual clarity even about the definition of the derivative. Thus, defining the derivative as a limit forces a differentiable function to be continuous. However, in physics it is routinely necessary to differentiate discontinuous functions. The Schwartz derivative (which needs the Lebesgue integral) permits this, but it does *not* generalize the classical definition.[20] Now, science must be refutable, so we cannot switch definitions as convenient, but must stick to one definition. But neither definition of derivative can be used either in classical physics (where the differential equations are nonlinear[21]) or in quantum field theory[22]—in both cases we need to differentiate and multiply. With the derivative as limit we cannot differentiate a discontinuous function, with the Schwartz theory we cannot multiply distributions.[23]

So, how should calculus be taught? First, it is clear that if there is no universal notion of rigor, calculus ought not to be taught for "rigor". On the contrary, now that the religious bias in the Western notion of rigor has been clearly pointed out, it would be unconstitutional in many countries to continue to teach calculus (or math) that way. For schools and undergraduate course, it should only be taught for its practical applications. These typically involve the numerical solution of various sorts of differential equations, today obtained using digital computers.

This was, in fact, the way the calculus actually originated in India.[24] Thus, the two key means of wealth in India were agriculture and overseas trade.[25] Indian monsoon-driven agriculture requires an accurate calendar,[26] which, in turn, needs accurate astronomical models. Likewise, the vast Indian overseas trade needed accurate navigation. Both required accurate trigonometric values. These (the finite

differences, actually) were calculated in a novel way by Aryabhata[27] in the 5th c. by numerically solving the ordinary differential equations for sine and cosine by what is today called "Euler's" method.[28] (Euler, of course, studied the Indian calendar and wrote on it.)

Aryabhata's elegant numerical technique easily led to 24 trigonometric values precise to 5 decimal places over the entire arc. Over the next thousand year, this technique was developed by mathematicians and astronomers of the Aryabhata school in Kerala (wrongly called the Kerala school), who used infinite series expansions to obtain trigonometric values precise to 9 decimal places.[29]

Precise trigonometric values were then badly required for the solution of the European navigation problem (specific to Europeans who used maps and lacked a technique of celestial navigation).[30] This was the major European scientific problem of the time, and large prizes were offered for its solution by various European governments. The key difficulties then were to determine latitude, longitude and loxodromes.[31] Navigational theorists like Simon Stevin used Aryabhata's values from Arab sources.[32] Later, more accurate values were transmitted direct from India in the 16th c. by Jesuits such as Matteo Ricci,[33] and others in the Jesuit college located in Cochin, in the vicinity of the Aryabhata school in Kerala. These Jesuits translated Indian texts, in Toledo mode, and sent them back to Europe. Those were first used by Mercator, then Christoph Clavius,[34] Tycho Brahe and his assistant Kepler, they went on to Galileo, and his student Cavallieri, and Fermat and Pascal,[35] and then diffused throughout Europe.

Europeans then (and now) failed to understand how Indians summed the infinite series. Because they mixed math with religious beliefs, they insisted that math must be perfect. Hence, Descartes[36] stated that summing an infinite series, to obtain the number π, was beyond the human mind. Thus, physically summing the entire series would take an eternity of time, while stopping at any point would mean an approximation, which was not perfect, hence not mathematics. Galileo concurred, in his letters to Cavalieri. Newton thought[37] he had resolved the difficulty with fluxions (which only made time metaphysical,[38] and hence led to the failure of his physics[39]). Berkeley attacked Newton,[40] and the story from that point to the present-day is well known.

Now, a formal description of the Indian technique of summing infinite series would be that they did order counting[41] by discarding infinitesimals in the field of rational functions, *larger* than formal reals. (Rational functions were treated as analogous to rational numbers.) But, what is today called the "Archimedean property" fails in a such a field, so that limits are not unique, and infinitesimals must be discarded. Note that we are speaking of an actual field extension of formal reals, not an intermediate[42] non-standard extension. Naturally, Indians did not suppose like Europeans that there was some sort of God-given notion of infinitesimal. In fact, this "discarding of infinitesimals" involved a sophisticated philosophy which I have called zeroism.

Zeroism takes off from Buddhist *sunyavada*, but for its practical benefits, and *without* reference to any Buddhist texts, like those of Nagarjuna.[43] It is a realistic philosophy which regards idealism as *erroneous*. In particular, the idea of mathematics as perfect, which is at the root of European epistemological difficulties with the calculus, stands rejected.

The issue of discarding small ("infinitesimal") quantities (as in summing the infinite series for π, or in the frequentist interpretation of probability) now appears as a problem of *representation,* which is readily resolved. Incidentally, zeroism also resolves the problem of the frequentist interpretation,[44] which cannot be resolved by conventional limits.

Calculus has been successfully taught with this new philosophy over the last two years, in teaching experiments conducted in universities in India and Malaysia.[45] Eliminating the theology from math makes math easy. This has been demonstrated by teaching calculus very quickly (typically in 5 days) even for non-math students (who did not do any advanced math in school). The post-test included a randomized selection of problems from published calculus question banks. Other demonstrated advantages of the new pedagogy of the calculus include the ability of school students to do advanced modelling (such as the amplitude-dependent motion of the simple pendulum, typically done through Jacobian elliptic functions).[46]

Reports of the new history and philosophy of math, and these pedagogical experiments have already been widely published in scholarly articles, books,[47] blogs,[48] and media.[49]This paper aims to bring home these ideas directly to ICME participants.

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2 W. W. Rouse Ball, *A Short Account of the History of Mathematics*, Dover, New York, 1960, pp. 1–2.

3 C. K. Raju, "Teaching mathematics with a different philosophy. 1: Formal mathematics as biased metaphysics." *Science and Culture*, **77** (7-8) pp. 274-79.

4 C. K. Raju, "Towards Equity in Math Education 1. Good-Bye Euclid!", *Bharatiya Samajik Chintan* **7 (**4) (New Series) (2009) pp. 255–264. C. K. Raju, "Teaching racist history", *Indian Journal of Secularism* **11**(4) (2008) 25-28.C. K. Raju, *Is Science Western in Origin?* Multiversity, Penang, 2009. C. K. Raju, talk on "Decolonising History: Goodbye Euclid!", chaired by Malaysian Deputy Minister of Higher Education, video at http://web.usm.my/webstreaming/flv/colonised.asp

5 "Euclid and Hilbert", chp. 1 in C. K. Raju, *Cultural Foundations of Mathematics*, Pearson Longman, 2007. Also, C. K. Raju, "How Should Euclidean Geometry be Taught", in Nagarjuna G. (ed) *History and Philosophy of Science: Implications for Science Education*, Homi Bhabha Centre, Bombay, 2001, pp. 241–260..

6 See C. K. Raju, "Euclid and Hilbert" cited above.

7 Plato, *Republic*, Book VII, 526 et seq., trans. J. L. Davies and D. J. Vaughan, Wordsworth, Hertfordshire, 1997, p. 240. Also, Jowett's translation (cited above, p. 394).

8 Proclus, *Commentary*, trans. Glenn R. Morrow, Princeton University Press, Princeton, 1992, 47, p. 38.

9 C. K. Raju, *Euclid and Jesus,* submitted for publication.

10 Plato, *Meno*, in *Dialogues of Plato*, trans. B. Jowett, Encyclopaedia Britannica, Chicago, 1996, pp. 179–180.

11 For an expository account, see C. K. Raju, *The Eleven Pictures of Time*, cited above. For specific quotations from Origen's *De Principiis*, see C. K. Raju, "Appendix on Origen", at http://ckraju.net/papers/Appendix-on-Origen.pdf

12 C. K. Raju, "The curse on 'cyclic' time", chp. 2 in *The Eleven Pictures of Time*, Sage, 2003.

13 C. K. Raju, "The Religious Roots of Mathematics", *Theory, Culture & Society* **23**(1–2) Jan-March 2006, Spl. Issue ed. Mike Featherstone, Couze Venn, Ryan Bishop, and John Phillips, pp. 95–97.

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15 C. K. Raju,"Logic". Article for *Springer Encyclopedia of Non-Western Science, Technology and Medicine,* Springer, 2008. http://ckraju.net/papers/Non-western-logic.pdf. Also, article on "Time" in the same.

16 For a technical exposition see "Quantum-mechanical time", chp. 6B in C. K. Raju, *Time: Towards a Consistent Theory*, Kluwer Academic, Dordrecht, 1994. Available from arxiv.org: 0808.1344. An account for the layperson is provided in C. K. Raju, *The Eleven Pictures of Time*, Sage, 2003. The connection to Buddhist logic is in C. K. Raju, "Probability in Ancient India", chp. 37 in *Handbook of the Philosophy of Science*, vol 7. *Philosophy of Statistics*, ed. Prasanta S. Bandyopadhyay and Malcolm R. Forster. General Editors: Dov M. Gabbay, Paul Thagard and John Woods. Elsevier, 2011, pp. 1175-1196.

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19 C. K. Raju, *Ending Academic Imperialism: a Beginning,* Multiversity, Penang, 2011. Also, "Ending Academic Imperialism in Hard Sciences: a Beginning", in *Confronting Academic Knowledge*, ed. Sue-San Gahremani Ghajar and Seyyed-Abdolhamid Mirhosseini, Iran Universities Press, Tehran, 2011, chp. 7. A longer draft at http://multiworldindia.org/wp-content/uploads/2010/05/ckr-Tehran-talk-on-academic-imperialism.pdf.

20 W. Rudin, *Functional Analysis*, McGraw Hill, New Delhi, 1974, p. 144. Since distributions lack point values, there are other ways in which this definition fails to generalize the classical definition.

21 C. K. Raju, "Junction Conditions in General Relativity." *J. Phys. A: Math. Gen.* **15** (1982) pp. 1785–97

22 C. K. Raju, "On the square of x^{-n}", *J. Phys. A: Math. Gen.* **16** (1983) pp. 3739–53.

23 For a quick summary of how the infinities of quantum field theory relate to shocks and to products of distributions, see C. K. Raju, "Distributional matter tensors in relativity". *Proceedings of the Fifth Marcel Grossmann Meeting on General Relativity*, ed. D. G. Blair and M. J. Buckingham, World Scientific, Singapore, 1989, pp 419–22.. Arxiv.org:0804:1998v1. For more details, see C. K. Raju, "Renormalization and shocks", appendix to *Cultural Foundations of Mathematics*, cited earlier.

24 C. K. Raju, *Cultural Foundations of Mathematics. The nature of mathematical proof and the transmission of the*

calculus from India to Europe in the 16th c. CE, Pearson Longman, 2007. For a quick summary, see *Ghadar Jari Hai*, 2(1) (2007) pp. 26–29. Full text: http://IndianCalculus.info/Cultural-Foundations-Mathematics-review-GJH.pdf.

25 For a quick summary, see *Ghadar Jari Hai*, 2(1) (2007) pp. 26–29. Full text: http://IndianCalculus.info/Cultural-Foundations-Mathematics-review-GJH.pdf.

26 This need still persists. See the recent newspaper clips over 3 years about the "delayed monsoon" and subsequent floods at http://ckraju.net/usm/math-tech.pdf. The phenomenon is described in Cultural Foundations of Mathematics, chp. 3, one part available online at "Could India's 'failed' monsoon have been predicted by the right calendar?", http://www.infinityfoundation.com/mandala/t_es/t_es_raju_monsoon_frameset.htm.

27 Aryabhata, *Gitika* 10. *Aryabhatiya of Aryabhata*, trans. K. S. Shukla and K. V. Sarma, INSA, New Delhi, 1976

28 For details, see C. K. Raju, "Infinite series and π", chp. 3 in *Cultural Foundations of Mathematics*, cited above.

29 *Cultural Foundations of Mathematics*, chp. 3, table 3.2 accuracy of Madhava's sine table, p. 122.

30 C. K. Raju, "Kamal or Rapalagai", *Cultural Foundations of Mathematics*, chp. 5. Also, earlier version of 1998 in *Indo-Portuguese Encounters* (ed) L. Varadarajan, INSA, New Delhi, Universidade Nova de Lisboa, Lisbon, 2006, vol 2, pp. 483–504.

31 C. K. Raju, :How and Why the Calculus was Imported into Europe", chp. 7 in *Cultural Foundations of Mathematics,* cited above. For a summary account see, For a quick account, see, C. K. Raju, "The Infinitesimal Calculus: How and Why it was Imported into Europe." Talk delivered at the International Conference on *Knowledge and East-West Transitions*, National Institute of Advanced Studies, Bangalore, Dec 2000. Abstract at http://ckraju.net/IndianCalculus/Bangalore.pdf.

32 *The Principal Works of Simon Stevin*, Vol. III. *Astronomy and Navigation*, ed. A Pannenkoek and Ernst Crone, Swets and Seitlinger, Amsterdam, 1961.

33 Letter by Matteo Ricci to Petri Maffei on 1 Dec 1581. *Goa* **38** I, ff 129r–30v, corrected and reproduced in *Documenta Indica*, XII, 472-477 (p. 474). Also reproduced in Tacchi Venturi, *Matteo Ricci S.I., Le Lettre Dalla Cina* 1580–1610, vol 2, Macareta, 1613.

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35 C. K. Raju, :How and Why the Calculus was Imported into Europe", chp. 7 in *Cultural Foundations of Mathematics,* cited above.

36 René Descartes, *The Geometry*, trans. David Eugene and Marcia L. Latham, Encyclopaedia Britannica, Chicago, 1996, Book 2, p. 544.

37 Newton provided this account anonymously, in his review of his own investigative report on Leibniz's charge of plagiarism against him.. Isaac Newton, "An account of the book entituled Commercium epistolicum collinii Et aliorum, de analysi promota", *Philosophical Transaction of the Royal Society of London*, No. 342. January and February 1714/15, pp. 173–224.

38 C. K. Raju, "Time: What is it that it can be Measured", *Science & Education* **15** (2006) pp. 537-551.

39 C. K. Raju, *Time: Towards a Consistent Theory*, Kluwer Academic, Dordrecht, 1994. C. K. Raju, "Retarded Gravitation Theory: the Gravitational Velocity Effect in Galactic Rotation Curves, and the Flyby Anomaly". arXiv:1102.2945v2.

40 George Berkeley, *The Analyst or a Discourse Addressed to an Infidel Mathematician*, ed. D. R. Wilkins, 1734, available online at http://www.maths.tcd.ie/pub/HistMath/People/Berkeley/Analyst/Analyst.html. For a full discussion of these objections, see C. K. Raju, "Numbers, Computers, and Algorismus", in: *Cultural Foundations of Mathematics*, cited earlier.

41 *Cultural Foundations of Mathematics*, cited earlier, chp. 3.

42 K. D. Stroyan and W. A. J. Luxembourg, *Introduction to the Theory of Infinitesimals*, Academic Press, New York, 1976.

43 C. K. Raju, "Numbers in calculus..." cited above. Also, "Zeroism and calculus without limits," paper presented at the 4th Nalanda Dialogue, 2008. Available from http://ckraju.net/papers/Zeroism-and-calculus-without-limits.pdf.

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45 C. K. Raju, "Teaching mathematics with a different philosophy. 1: Formal mathematics as biased metaphysics." *Science and Culture*, **77** (7-8) pp. 274-79. and "2: Calculus without limits." *Science and Culture*, **77** (7-8) pp. 280-86.

46 C. K. Raju, "Time: What is it that it can be Measured", *Science & Education* **15** (2006) pp. 537-551.

47 Apart from the books cited above, this includes *Making math easy* (submitted for publication).

48 See, for example, "Decolonising our Universities: Time for Change", http://globalhighered.wordpress.com/2011/09/11/decolonising-our-universities-time-for-change/. This and other links are available also at this author's personal blog http://ckraju.net/blog. This includes a fairly extensive archive of the public

debate on decolonisation, as well as the issues earlier taken up with the National Knowledge Commission in India.

49 Media reports include the 2-page *New Strait Times* report on "Mathematics Made Easy": http://www.nst.com.my/nst/articles/ISSUES_Mathematics_Noproblem/Article/, and the earlier 2-page report on "Reimagining the University". News clips of this and other news reports together with the long (3-month) discussion in the press, on issues arising from the decolonisation conference, is archived at http://ckraju.net/blog/?p=62, and http://ckraju.net/blog/?p=61.