Calculus without Limits: the Theory

A Critique of Formal Mathematics
Part 1: Axioms and Definitions

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Paradoxes of set theory

Why $\mathbb{R}$?

How to define the derivative?

Conclusions

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Why are limits important?

- We saw that it is impossible to teach limits?

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- limits are taught for rigor.

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- The rigorous approach to calculus—also called mathematical analysis—allows us to *prove* the existence and uniqueness of limits.
- The mathematician believes this answer, and other persons in the community of mathematicians may share this belief.
- But how far is it true?

What is rigor actually?

- I will argue that rigor = reliance on the arbitrary decisions of those in mathematical authority.

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- Today I look at arbitrariness in axioms and definitions,
- to demonstrate the arbitrariness in calculus from within formal mathematics.

Set theory and supertasks

- Historically, the construction of $\mathbb{R}$ by Dedekind cuts involved Cantor's set theory.

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- After the calculus came to Europe (in the 16th c.) there were epistemic doubts about its validity.
- Mathematicians thought of summing an infinite series by actually carrying out the sum,
- and this seemed a supertask (an infinite series of tasks).

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- Even the simplest set theoretic statement "let $x \in \mathbb{R}$" involves a supertask.
- This involves the claim that it is possible to select and specify a real number, in a way that singles it out *uniquely* from an infinity of adjacent real numbers.
- This is a supertask.

An example

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- The fastest computers today can manage teraflops, or around 10^{12} floating point additions per second.
- If we use this computer exclusively to add continuously for a year: we can only go up to 10^{20} additions—still a long way from infinity.

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- this is not a task which is physically every going to be possible.
- But set theory allows us to do it metaphsyically.
- In fact, set theory allows us to specify an uncountable infinity of real numbers!

A note

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- which has a very old history, dating back to times when European culture had not even begun.
- Such approximations are readily possible
- the question of a supertask arises only when we speak of being able to specify the value of π exactly or uniquely.

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- got pushed into doubts about set theory.

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Dedekind's real achievement

- The use of $\mathbb{R}$ for calculus means that doubts about supertasks,
- which were earlier attached to the calculus,
- got pushed into doubts about set theory.
- From this perspective, Dedekind's real achievement was that he pushed doubts about supertasks and infinity away from nubers and into the domain of set theory.

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- They can say that it is not their job ("proof by territory limitation").
- they can say (as Paul Erdos nearly said), "so many people believe it, they can't all be wrong can they?" (proof by numbers"), etc.
- (For more details about such proofs, see the appendix to my book The Eleven Pictures of Time, Sage, 2003.)

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- Thus, with $\mathbb{R}$, doubts about supertasks in the calculus were pushed out of what mathematicians regard as their normal area of activity,
- and into set theory.
- But were they resolved?

Traditional paradoxes of infinity

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- A classic example is the Sanskrit śloka, the first verse of the Īśā Upaniśad, "ॐ पूर्णमदः पूर्णमिदम पूर्णात् पूर्णमुदच्यते / पूर्णस्य पूर्णमादाय पूर्णमेवावशिष्यते".

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- The second line says "if you remove the whole from the whole, what remains is the whole".
- It was such paradoxes which made Descartes and Galileo suspect the calculus when it first arrived in Europe (as we will see in more detail, later on).

Paradoxes of infinity in the Christian tradition

- A similar paradox was encountered in Christian tradition a thousand years before Descartes.

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- John Philoponus, in his *Apology Against Proclus*, defended the idea that the world was created,
- He argued that adding a day to eternity would not change eternity. Hence, the world was not eternal.
- Curiously, he had a different attitude towards the eternal torture in hell which he thought awaited non-Christians, a torture which he thought they would experience for an eternity of time.

The double standard

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The double standard

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- but this is much harder to spot.
- Let us try.

Russell's paradox

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- Recall Russell's paradox.
- Let $R = \{x | x \notin x\}$.
- Now, if $R \notin R$, then, by definition, we must have $R \in R$.
- On the other hand, if $R \in R$ then, again, by definition, we must have $R \notin R$.
- So, either way, we have a contradiction.

The definition of a set

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The definition of a set

- The way these contradictions are resolved in axiomatic set theory is peculiar.
- Take, for example, the von-Neumann-Bernays-Gödel (NBG) set theory.
- Here, a well-formed formula (of the sort used in Russell's paradox) in general only defines a class.
- A set is defined as a class A for which $\exists$ a class B, such that $A \in B$.

Resolution of Russell's paradox in NBG

- Russell's paradox is resolved in NBG by saying that the Russell class is a class, not a set, for we cannot find a class S such that $R \in S$.

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- Russell's paradox is resolved in NBG by saying that the Russell class is a class, not a set, for we cannot find a class S such that $R \in S$.
- The paradoxes of set theory apply to classes, not sets.
- Mathematicians can stick to sets and thus avoid the paradoxes which are now (believed to be) confined to classes.

So are all paradoxes resolved?

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So are all paradoxes resolved?

- How can we be sure that all paradoxes of set theory are resolved.
- NBG includes classes which are paradoxical.
- How can we be sure this does not make NBG inconsistent?

The consistency of NBG

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The consistency of NBG

- The consistency of NBG is not proven
- it is only widely believed among mathematicians.
- So, basing the calculus on $\mathbb{R}$ and NBG does not guarantee the surety of the results.
- That's only a belief.

Metamathematics

- It is interesting to see how the belief in the consistency of NBG is maintained.

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Metamathematics

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- By Gödel's second incompleteness theorem, the consistency of a consistent theory cannot be proven within the theory.
- Therefore, to decide the consistency of set theory we require metamathematics.
- The question is: what kind of metamathematics?
- Before answering this question, let us recall some socially accepted results of metamathematics.

Cantor's Continuum Hypothesis

- For a set X denote its cardinality by $\#(X)$.

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- Not clear what happens when X is infinite.
- Recall that Cantor's continuum hypothesis states that if $\aleph_0$ is the cardinality of the infinite set $\mathbb{N}$ of natural numbers, and c is the cardinality of $\mathbb{R}$ then $2^{\aleph_0} = c$.

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- Not clear what happens when X is infinite.
- Recall that Cantor's continuum hypothesis states that if $\aleph_0$ is the cardinality of the infinite set $\mathbb{N}$ of natural numbers, and c is the cardinality of $\mathbb{R}$ then $2^{\aleph_0} = c$.
- The metamathematical theorems of Gödel and Cohen showed that the continuum hypothesis (CH) implies (but is not implied by) the axiom of choice.

Axiom of Choice

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- A choice function f for a set X allows us to pick an individual element $f(A) \in A$ for each $A \in X$.

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- A choice function f for a set X allows us to pick an individual element $f(A) \in A$ for each $A \in X$.
- Equivalent is Zorn's Lemma: in a partially ordered set if every chain is bounded above, then there must be at least one maximal element,
- or Hausdorff maximality principle: in a partially ordered set every chain is contained in a maximal chain etc.

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contd.

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- These are today part of the everyday equipment of mathematical reasoning.
- The AC is needed to prove what are regarded as everyday results today:
- the existence of a Lebesgue non-measurable set or Tychonoff's theorem (that the product of compact sets is compact) etc.
- Zorn's lemma is used to prove the Hahn-Banach theorem etc.

Banach-Tarski paradox

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Banach-Tarski paradox

- However, the AC (and the existence of Lebesgue non-measurable sets) also leads to the Banach-Tarski paradox.
- Namely, let $A, B \subset \mathbb{R}^n$, with $n \geq 3$.
- Further, let A, B be bounded and have non-empty interior.

Banach-Tarski paradox

- However, the AC (and the existence of Lebesgue non-measurable sets) also leads to the Banach-Tarski paradox.
- Namely, let $A, B \subset \mathbb{R}^n$, with $n \geq 3$.
- Further, let A, B be bounded and have non-empty interior.
- Then, there exist finite partitions of A, B, such that $A = \bigcup_{i=1}^{k} A_i$, $B = \bigcup_{i=1}^{k} B_i$, and each A_i is congruent (under Euclidean motions) to B_i.

Banach-Tarski Paradox

contd

- This paradox conflicts violently with geometric intuition, for it means that a ball in 3-dimensional space may be broken into a finite number of non-overlapping pieces, which may be reassembled by rotation and translation (without stretching) into *two* balls of the same volume as the original.

The theorems of Gödel and Cohen

- such paradoxes created fears that AC may lead to inconsistency of NBG.

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- We look at the formal contrapositive:
- if set theory is inconsistent with AC, then it must be inconsistent without AC.

What kind of metamathematics

- To return to the original question.

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- Metamathematics needed to prove consistency of NBG,
- But what kind of metamathematics?
- Specifically, can principles like AC and CH be admitted in metamathematics?

Deciding decidability

- By Gödel's first incompleteness theorem, any formal theory large enough to contain natural numbers contains a proposition asserting its own negation

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- which cannot *hence* be either proved or disproved within the theory (if the theory is consistent; if it is inconsistent, every statement is provable).
- However, if such a theory is decidable, then the statement *can* be either proved or disproved within the theory.
- That is, if set theory is decidable it must be inconsistent.

Is set theory consistent?

- Decidability of a formal theory is usually understood in the sense of recursive decidability.

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- But, why should we limit metamathematics to finite recursion?
- Conjecture: Transfinite recursion (an easy consequence of AC), makes set theory decidable (hence inconsistent).

Is set theory consistent?

contd.

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- And if we find it trustworthy, why not allow it also in metamathematics?
- So, standard of proof in metamathematics $\neq$ standard of proof in mathematics. Why?
- The only answers is from mathematical authority. So formal mathematics ultimately depends upon authority, not reason.

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- Infinite procedures (even AC) allowed for proofs in mathematics, but disallowed in metamathematics.
- This is a hypocritical social consensus among authoritative Western mathematicians. Ideally, there should be one standard of proof for both mathematics and metamathematics.

Completeness of $\mathbb{R}$

- Why is $\mathbb{R}$ needed for calculus?

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- Conventional answer: because $\mathbb{R}$ is complete (as a metric space).

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- so the difference between the m^{th} and n^{th} term can be made less than 10^{-q} where $q = \min\{m, n\}$.

Completeness of $\mathbb{R}$

- However, this Cauchy sequence does not converge in $\mathbb{Q}$ since $\mathbb{Q}$ is not complete.

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- From the construction of $\mathbb{R}$ as the set of equivalence classes of Cauchy sequences in $\mathbb{Q}$, this does not happen in $\mathbb{R}$ which is complete.
- What happens in a field larger than $\mathbb{R}$?

Archimedean Property

- $\mathbb{R}$ has the Archimedean property (AP).

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- AP characterizes $\mathbb{R}$. That is, $\mathbb{R}$ is the largest ordered field with AP.
- Consequently, if we have an ordered field $\mathbb{S} \supset \mathbb{R}$, then the AP must fail in $\mathbb{S}$.

Infinities and infinitesimals in an ordered field

- Such a field $\mathbb{S}$ in which the AP fails, must have both infinities and infinitesimals.

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- Further, since $\mathbb{S}$ is an ordered field, this x must have a multiplicative inverse $\frac{1}{x}$. This must satisfy $0 < \frac{1}{x} < \frac{1}{n}$ for all $n \in \mathbb{N}$.

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- Thus, $\frac{1}{x}$ corresponds to what we intuitively understand as an infinitesimally small number.

Limits in a field without AP

- What would happen to limits in such a field?

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- Note: we are here *not* talking about non-standard analysis: the infinities and infinitesimals in the field $\mathbb{S}$ do not arise merely at an intermediate stage: they are "permanent", so to say.

Example of an ordered field without AP

- This example also required for later philosophy of zeroism.

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- Since $\mathbb{R}$ is a field, it is well known P must be an integral domain.

example of an ordered field without AP

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- The formal quotient, such as $\frac{x-2}{x-3}$ is defined whenever the denominator is a non-zero polynomial, even though, as a *function*, it may be infinite (or fail to be defined) at a finite set of points (the roots of the denominator).
- To avoid quibbles concerning the form $\frac{0}{0}$, we can define two rational functions to be equivalent if they differ only on a finite set of points. (This can happen also with equivalent formal quotients, e.g. $\frac{x(x-1)}{x-1}$ and $\frac{x(x-2)}{x-2}$.)

Interim summary

- As we will see in more detail later on, this is how the calculus originally developed in India.

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- and it was acceptable that limits are not unique.
- Right now the question is only this: why do calculus in $\mathbb{R}$? why not use such an $\mathbb{S}$ which makes calculus easier and more intuitive?
- The only answer is that conventional calculus teaching uncritically imitates the European historical experience of the calculus.

Inadequacy of the classical definition

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- With this definition a differentiable function must be continuous.
- So, a discontinuous function may not be differentiated.

The Dirac δ

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- The Dirac δ had a sad childhood:
- physicists denied that it was physical, and used it as purely a mathematical artifice.
- Mathematicians, on the other hand, considered it as something non-mathematical and non-rigorous—a mere construct used by physicists.

Formalizations

- Heaviside, however, used it for electrical engineering.

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- Formally, this corresponds to the formula for integration by parts:

$$\int_{-\infty}^{\infty} f'g = -\int_{-\infty}^{\infty} fg'.$$

- Here, f is the function (possibly discontinuous) which one seeks to differentiate,
- and the derivative f' is now being defined by the right hand side, where the derivative is transferred to
- the test function g which is assumed to be infinitely differentiable: $g \in C^{\infty}$.

Test functions

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- and the above formula corresponds to the formula for integration by parts.
- This works equally well for functions of several variables, and we can write

$$\int_{\mathbb{R}^n} f'g = -\int_{\mathbb{R}^n} fg',$$

for $g \in D(\mathbb{R}^n)$ where $D(\mathbb{R}^n)$ is the space of compactly supported and infinitely differentiable functions.

The space of test functions

- Formally, $D(\mathbb{R}^n)$ is a topological vector space with the topology of uniform convergence on compacta to all orders.

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  - Here $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index, and $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$.

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  - These seminorms p_N generate a vector topology on $C^\infty(\mathbb{R}^n)$, in which the space of compactly supported test functions D is a closed subspace.

Which derivative?

- The Schwartz theory requires that the integral be the Lebesgue integral and not the Riemann integral.

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- with the Schwartz theory every integrable functions is differentiable.
- ϵ–δ definition of the limit and the corresponding derivative was not "natural".
- That was just a consensus among mathematicians, which has changed, because the earlier definition was not adequate for physics.

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contd

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- However, the Dirac delta is not the zero distribution, since $\int \delta(x)dx = 1$.
- Thus, for purposes of physics, we need to settle on one of the two as the right definition, and clearly the Schwartz definition is better than the older ϵ–δ definition.

Difficulty of point values and products

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- This loss of point values already occurred in the Lebesgue theory of integration.
- However, it has more serious consequences in the Schwartz theory.
- Pointwise products of functions are no longer defined.

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- Possible to give a natural-looking definition of the pointwise product when only one of the functions is C^∞.
- Called the Schwartz product. If g is a distribution, and $f \in C^\infty$, define

$$\langle fg, h\rangle = \langle g, fh\rangle$$

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  for all test functions h, where $\langle f, h\rangle \equiv \int fh$.
- If $f \in C^\infty$ and h is a test function, $f.h$ is again a test function. Hence, the right hand side is well defined.

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- (a) agrees with the Schwartz product (defined above),
- (b) is associative (that is $(fg)h = f(gh)$ for all distributions f, g, h), and
- (c) satisfies the Leibniz rule (that is $(fg)' = fg' + f'g$ for all distributions f, g).

Taub's remark

- Taub[1] asserted, "Fortunately, the product of such distributions [as arise] is quite tractable".

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$$2\theta \cdot \theta' = \theta',$$
- Since $\theta' = \delta$, this can be rewritten as
$$2\theta \cdot \delta = \delta,$$
which immediately tells us that
$$\theta \cdot \delta = \frac{1}{2} \cdot \delta.$$

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- Comparing the above two leads to the interesting conclusion that $\frac{1}{2} = \frac{1}{3}$!

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- Products of these propagators arise in the S-matrix expansion.
- These products are Fourier transformed into convolution integrals, which are divergent.
- If we apply this to δ^2 we see that

$$\widehat{(\delta^2)} = \hat{\delta} * \hat{\delta} = 1 * 1 = \int 1 = \infty.$$

Arbitrariness in the definition of the product

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- Quantum field theorists use the Hahn-Banach definition useless for classical physics (shock waves).
- Mathematicians use Colombeau's product useless for physics (since it is both associative and satisfies the Leibniz rule).
- What are the principles on which the choice is to be decided?

Which definition of the product?

- One possibility is to use comparison theorems.

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- This is decided by "other considerations". Colombeau product exactly like naive product of non-standard distributions.
- Since associate law and Leibniz rule holds, it has a problem as follows.

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- With the Colombeau theory, it is not possible to discriminate between forms (a) and (b).

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- Calculus with limits is taught on grounds of rigor. However, this purported rigor depends upon the imposition of a variety of arbitrary choices.
  - **The choice of metamathematics is arbitrary**. Calculus with limits requires infinite procedures (spertasks), incorporated in $\mathbb{R}$ which is constructed using axiomatic set theory, such as NBG. Supertasks lead to paradoxes of set. Consistency of NBG can only be proved or disproved in metamathematics. The consistency is maintained by an arbitrary choice of metamathematics: refusing to allow in metamathematics the sort of infinite procedures for proof that are admitted in NBG.

Conclusions

contd

- **The choice of the number system underlying the calculus is arbitrary.** It is possible to do calculus more intuitively in non-Archimedean fields larger than $\mathbb{R}$.

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- **The choice of the number system underlying the calculus is arbitrary.** It is possible to do calculus more intuitively in non-Archimedean fields larger than $\mathbb{R}$.
- **The definition of the derivative is arbitrary**. The classical ϵ–δ definition of the derivative is not adequate for physics, since the derivative of discontinuous functions naturally arises in physics.

Conclusions

contd

- **The definition of the product of distributions is arbitrary** The classical definition of derivative is usually replace by the Schwartz definition which is incomplete since it does not address the issue of products of distributions. Colombeau's simplistic definition is today being promoted by mathematical authority, although it is inadequate and inappropriate for physics

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- **The definition of the product of distributions is arbitrary** The classical definition of derivative is usually replace by the Schwartz definition which is incomplete since it does not address the issue of products of distributions. Colombeau's simplistic definition is today being promoted by mathematical authority, although it is inadequate and inappropriate for physics
- As seen by the fate of the classical definition of derivative, ultimately mathematical definitions have to be related to practical value not mathematical authority.